NOTES ON DIFFERENTIAL GEOMETRY

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Part 1. Geometry of Curves

We assume that we are given a parametric space curve of the form

$$\mathbf{x}(u) = \begin{bmatrix} x_1(u) \\ x_2(u) \\ x_3(u) \end{bmatrix} \qquad u_0 \le u \le u_1 \tag{1}$$

and that the following derivatives exist and are continuous

$$\mathbf{x}'(u) = \frac{d\mathbf{x}}{du} \qquad \mathbf{x}''(u) = \frac{d^2\mathbf{x}}{du^2} \tag{2}$$

1. Arc Length

The total arc length of the curve from its starting point $\mathbf{x}(u_0)$ to some point $\mathbf{x}(u)$ on the curve is defined to be

$$s(u) = \int_{u_0}^{u} \sqrt{\mathbf{x}' \cdot \mathbf{x}'}\, du \tag{3}$$

It is also common to express this equation in a differential form:

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} \tag{4}$$

The differential ds is referred to as the *element of arc* of the curve.

Because we know that $ds/du \neq 0$, it is always permissible to reparameterize the curve $\mathbf{x}(u)$ in terms of its arc length $\mathbf{x}(s)$. This reparameterized curve has derivatives:

$$\dot{\mathbf{x}}(s) = \frac{d\mathbf{x}}{ds} \qquad \ddot{\mathbf{x}}(s) = \frac{d^2\mathbf{x}}{ds^2} \tag{5}$$

Such a parameterization of the curve is often called a *unit-speed parameterization* because $\|\dot{\mathbf{x}}\| = 1$.

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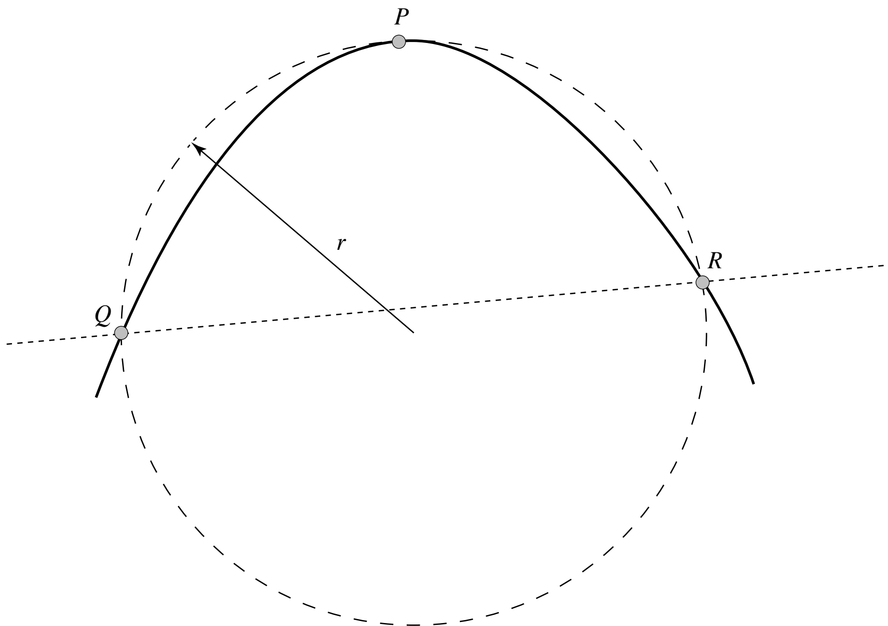


Figure 1. Local geometry around a point P. Points Q and R are equidistant from P along the curve.

2. Local Frames and Curvature

To proceed further, we need to more precisely characterize the local geometry of a curve in the neighborhood of some point. All the necessary properties of the curve can be derived algebraicly, as with the definition of arc length. However, before examining these algebraic definitions, let us consider a more direction construction that will provide greater intuition about the geometry of the curve.

2.1. **Geometric Construction.** Consider a point P on the curve, with additional points Q and R equidistant from P in opposite directions along the curve (see Figure 1). We can define a unique circle C passing through these points.

Now consider the circle C in the limit as Q and R approach P. This is called the *osculating circle* of the curve at P. It will pass through the point P, thus touching the curve at this point. The tangent line of C at P will also be the *tangent line* of the curve at P. Furthermore, the vector from P to the origin of C is obviously perpendicular to the tangent line at P, and is therefore a *normal vector* of the curve at this point. The circle C also has some radius ρ. We define the *curvature* at the point P to be $\kappa = 1/\rho$.

2.2. **Algebraic Definitions.** We assume that we are given a unit-speed parameterization (§1) of a curve $\mathbf{x}(s)$. The *unit tangent vector* $\mathbf{t}$ is simply the first derivative of $\mathbf{x}$:

$$\mathbf{t} = d\mathbf{x}/ds = \dot{\mathbf{x}} \tag{6}$$

Note that this is a unit vector precisely because we have assumed that the parameterization of the curve is unit-speed. The second derivative $\ddot{\mathbf{x}}$ will be orthogonal to $\mathbf{t}$, and thus defines a normal vector. The length of $\ddot{\mathbf{x}}$ will be the curvature κ. Therefore, we can define both the *curvature normal* $\mathbf{k}$ and the *unit normal* $\mathbf{n}$ as:

$$\mathbf{k} = d\mathbf{t}/ds = \ddot{\mathbf{x}} = \kappa\mathbf{n} \tag{7}$$

Since we are typically interested in curves embedded in E^3, we can also define the *unit bi-normal*

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} \tag{8}$$

For curves embedded in E^3, these three unit vectors provide a complete orthonormal basis. They are often referred to collectively as the *moving* or *local trihedron.*

From the three vectors of the local trihedron, we can also define three canonical planes through the point $\mathbf{x}$

$$\text{Normal plane:} \quad (\mathbf{y}-\mathbf{x})\cdot\mathbf{t} = 0 \tag{9}$$

$$\text{Rectifying plane:} \quad (\mathbf{y}-\mathbf{x})\cdot\mathbf{n} = 0 \tag{10}$$

$$\text{Osculating plane:} \quad (\mathbf{y}-\mathbf{x})\cdot\mathbf{b} = 0 \tag{11}$$

It is the osculating plane that is usually of most interest to us, as it is the plane that locally contains the curve (i.e., plane curves lie entirely within their osculating plane, which is everywhere the same).

3. Frenet Formulas

At each point on the curve, we can define a local trihedron $(\mathbf{t}, \mathbf{n}, \mathbf{b})$. Unless the curve is a straight line, the trihedron will change as we move along the curve. This naturally leads us to investigate how exactly the trihedron changes along the curve. This is most succinctly expressed using the *Frenet Formulas*, which give the derivatives of the trihedron vectors:

$$\begin{array}{rclccc} d\mathbf{t}/ds & = & & \kappa\mathbf{n} & \\ d\mathbf{n}/ds & = & -\kappa\mathbf{t} & & +\tau\mathbf{b} \\ d\mathbf{b}/ds & = & & -\tau\mathbf{n} & \end{array} \tag{12}$$

This is often encountered in matrix–vector form as well:

$$\begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{13}$$

Note that we have introduced a new local quantity — the *torsion* τ. Just as curvature measures the change in the normal along the tangent direction, the torsion measures the change in the normal along the bi-normal direction.

Intuitively, the torsion measures the differential "twisting" of the trihedron around the curve.

Interpreted kinematically, the motion of the local trihedron can be seen as a differential translation ($d\mathbf{x} = \mathbf{t}\,ds$) combined with a differential rotation. The axis of this rotation, often referred to as the *vector of Darboux*, is simply

$$\mathbf{r} = \tau\mathbf{t} + \kappa\mathbf{b} \tag{14}$$

This allows us to rewrite the Frenet equations in the following way

$$\dot{\mathbf{t}} = \mathbf{r}\times\mathbf{t} \qquad \dot{\mathbf{n}} = \mathbf{r}\times\mathbf{n} \qquad \dot{\mathbf{b}} = \mathbf{r}\times\mathbf{b} \tag{15}$$

Also note that $\mathbf{r}$ can be seen as an angular velocity vector, and thus $\omega = \|\mathbf{r}\|$ is the speed with which the trihedron is rotating.

We can perform a Taylor expansion of $\mathbf{x}$ around some point $\mathbf{x}(s_0)$:

$$\mathbf{x}(s_0 + h) = \mathbf{x}(s_0) + h\dot{\mathbf{x}}(s_0) + \frac{h^2}{2}\ddot{\mathbf{x}}(s_0) + \cdots \tag{16}$$

which leads to the following approximation of $\mathbf{x}$ in the neighborhood of a point $\mathbf{x}(s_0)$:

$$\mathbf{x}(s_0 + h) \approx \mathbf{x}_0 + h\mathbf{t}_0 + \kappa_0\frac{h^2}{2}\mathbf{n}_0 + \kappa_0\tau_0\frac{h^3}{6}\mathbf{b}_0 \tag{17}$$

Part 2. Geometry of Surfaces

Let us assume that we are given a closed differentiable manifold surface M which has been divided into a set of patches. A given surface patch is defined by the mapping

$$\mathbf{x} = \mathbf{x}(u, v) = \begin{bmatrix} f_1(u, v) \\ f_2(u, v) \\ f_3(u, v) \end{bmatrix} \tag{18}$$

where (u, v) range over a region of the Cartesian 2-plane and the functions f_i are of class C^2. We shall be concerned with the surface in the neighborhood of a point $\mathbf{p} = \mathbf{x}(u_0, v_0)$. By convention, all functions of $\mathbf{x}$ and its derivatives are implicitly evaluated at (u_0, v_0).

4. The Tangent Plane

The partial derivatives of the patch function $\mathbf{x}$

$$\mathbf{x}_1 = \mathbf{x}_u = \partial\mathbf{x}/\partial u \quad \text{and} \quad \mathbf{x}_2 = \mathbf{x}_v = \partial\mathbf{x}/\partial v \tag{19}$$

span the tangent plane of the surface at $\mathbf{p}$, provided we make the standard assumption that $\mathbf{x}_1 \times \mathbf{x}_2 \neq 0$.

Let $\mathbf{t}$ be a vector tangent to the surface $\mathbf{x}$ at the point $\mathbf{p}$. We know that we can write it as a linear combination $\mathbf{t} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$. Therefore, we can also conveniently represent this tangent vector as a direction vector $\mathbf{u} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

We will generally be concerned with *differential* tangent vectors, that we will write as:

$$d\mathbf{x} = du\, \mathbf{x}_1 + dv\, \mathbf{x}_2 \tag{20}$$

Here we have a corresponding direction vector $\mathbf{u} = \begin{bmatrix} du \\ dv \end{bmatrix}$.

Given that we have two vectors spanning the tangent plane, we can also compute the local unit surface normal $\mathbf{n}$ at the point $\mathbf{p}$:

$$\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\|\mathbf{x}_1 \times \mathbf{x}_2\|} \tag{21}$$

5. First Fundamental Form

We would like to compute the (squared) length of a given differential tangent vector $d\mathbf{x}$. As with any other vector in E^3, we compute the squared length of this vector via the inner product $d\mathbf{x} \cdot d\mathbf{x}$. Expanding this inner product, we arrive at:

$$d\mathbf{x} \cdot d\mathbf{x} = du^2\, \mathbf{x}_1 \cdot \mathbf{x}_1 + 2du\, dv\, \mathbf{x}_1 \cdot \mathbf{x}_2 + dv^2\, \mathbf{x}_2 \cdot \mathbf{x}_2 \tag{22}$$

This quadratic form is called the *first fundamental form*. The classical notation for this quadratic form, dating back to Gauss, was the following:

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2 \tag{23}$$

For our purposes, it is more convenient to write this quadratic form using matrix/vector notation. Assuming that we are given a direction vector $\mathbf{u} = [du\ dv]^\mathsf{T}$, then we can write the first fundamental form as:

$$d\mathbf{x}\cdot d\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{G}\mathbf{u} \qquad \text{where } \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad \text{with } g_{ij} = \mathbf{x}_i\cdot\mathbf{x}_j \tag{24}$$

The matrix $\mathbf{G}$ is usually referred to as the *metric tensor*. Since the dot product is commutative, it is clear that $g_{ij} = g_{ji}$ and thus $\mathbf{G}$ is symmetric. We will also frequently have need to use the determinant of the metric tensor:

$$g = \det \mathbf{G} = g_{11}g_{22} - g_{12}^2 \tag{25}$$

Notice that our assumption that $\mathbf{x}_1 \times \mathbf{x}_2 \neq 0$ implies that $g \neq 0$.

We can also use the metric tensor to measure the angle between two tangent vectors. Suppose we are given two direction vectors $\mathbf{u}$ and $\mathbf{v}$. The angle θ between them is characterized by:

$$\cos\theta = \frac{\mathbf{u}^\mathsf{T}\mathbf{G}\mathbf{v}}{(\mathbf{u}^\mathsf{T}\mathbf{G}\mathbf{u})(\mathbf{v}^\mathsf{T}\mathbf{G}\mathbf{v})} \tag{26}$$

In the special case of the angle $\hat{\theta}$ between the two isoparametric lines this reduces to:

$$\cos\hat{\theta} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}} \qquad \sin\hat{\theta} = \sqrt{\frac{g}{g_{11}g_{22}}} \tag{27}$$

5.1. **The Jacobian.** The Jacobian matrix of our surface patch $\mathbf{x}$ is the matrix of partial derivatives:

$$\mathbf{J} = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{bmatrix} = \begin{bmatrix} \vdots & \vdots \\ \mathbf{x}_1 & \mathbf{x}_2 \\ \vdots & \vdots \end{bmatrix} \tag{28}$$

The metric tensor $\mathbf{G}$ can also be derived as the product of the Jacobian with its transpose:

$$\mathbf{G} = \mathbf{J}^\mathsf{T}\mathbf{J} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix} \mathbf{x}_1\cdot\mathbf{x}_1 & \mathbf{x}_1\cdot\mathbf{x}_2 \\ \mathbf{x}_1\cdot\mathbf{x}_2 & \mathbf{x}_2\cdot\mathbf{x}_2 \end{bmatrix} \tag{29}$$

The Jacobian also provides a convenient notation for connecting the differential tangent $d\mathbf{x}$ with its direction vector $\mathbf{u}$. Specifically, $d\mathbf{x} = \mathbf{J}\mathbf{u}$.

5.2. **Element of Area.** Because it measures lengths and angles, the first fundamental form is also the key to defining surface area. Suppose that we are given a region Ω on our surface patch $\mathbf{x}$. Its area is given by the integral

$$\iint_\Omega dA = \iint \sqrt{g}\, du\, dv \tag{30}$$

The differential $dA = \sqrt{g}\, du\, dv$ is referred to as the *element of area* of the surface.

6. Second Fundamental Form

Suppose that we wish to measure the change of the normal vector $\mathbf{n}$ in a given tangential direction: $-d\mathbf{x}\cdot d\mathbf{n}$. This is the *second fundamental form.* Like the first fundamental form, it also has a classical Gaussian notation:

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 \tag{31}$$

We will generally work with the second fundamental form in its matrix version:

$$-d\mathbf{x}\cdot d\mathbf{n} = \mathbf{u}^{\mathsf{T}}\mathbf{B}\mathbf{u} \qquad \text{where } b_{ij} = \mathbf{n}\cdot\mathbf{x}_{ij} = -\mathbf{n}_i\cdot\mathbf{x}_j. \tag{32}$$

As with the first fundamental form, we define the determinant of this tensor

$$b = \det\mathbf{B} = b_{11}b_{22} - b_{12}^2 \tag{33}$$

7. Surface Curvature

We have already seen in Section 2 how to define the curvature of a space curve. We can readily extend this definition to define the curvature of a surface as well.

Consider a point $\mathbf{p}$ on our surface patch. At this point we can compute a unit surface normal vector $\mathbf{n}$. Now suppose that we select an arbitrary unit tangent vector $\mathbf{t}$, with corresponding direction vector $\mathbf{u}$. There is a unique plane passing through $\mathbf{p}$ containing both the vectors $\mathbf{n}$ and $\mathbf{t}$. In the neighborhood of $\mathbf{p}$, this plane intersects the surface along some curve. This curve is a *normal section* of the surface (see Figure 2).

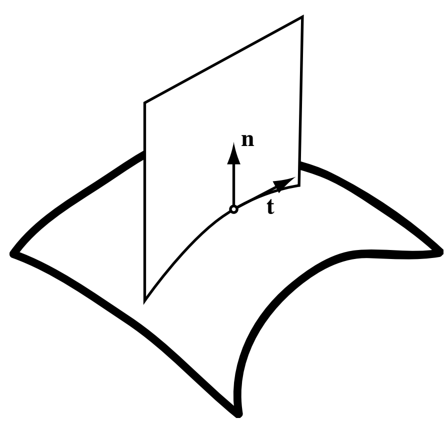


Figure 2. Normal section

Applying the constructions of Section 2, we can define the curvature of this normal section. We call this the *normal curvature* of the surface in the direction $\mathbf{u}$. We denote the normal curvature as κ_n. While technically normal curvature is a function of direction — its full form is $\kappa_n(\mathbf{u})$ — we generally drop the direction $\mathbf{u}$ for convenience. Its presence however is always implicit.

This definition of normal curvature is the most convenient intuitive defin-tion. We can also define it algebraically in terms of the fundamental forms. In particular, the normal curvature κ_n in the direction $\mathbf{u}$ is

$$\kappa_n = \frac{\mathbf{u}^{\mathsf{T}}\mathbf{B}\mathbf{u}}{\mathbf{u}^{\mathsf{T}}\mathbf{G}\mathbf{u}} \tag{34}$$

7.1. **Principal Curvatures.** Unless the curvature is equal in all directions, there must be a direction $\mathbf{e}_1$ in which the normal curvature reaches a maximum and a direction $\mathbf{e}_2$ in which it reaches a minimum. These directions are called *principal directions* and the corresponding curvatures κ_1, κ_2 are the *principal curvatures*. It also turns out that the normal curvature κ_n in an arbitrary direction can be written in terms of the principal curvatures:

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \tag{35}$$

where θ is the angle between the direction in question and the first principal direction.

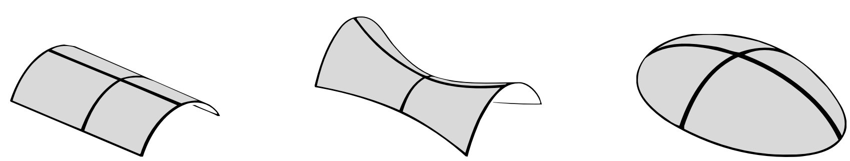
In addition to the principal curvatures, we are also interested in two important quantities. The first is the *mean curvature*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{36}$$

The second is the *Gaussian curvature*

$$K = \kappa_1\kappa_2 \tag{37}$$

A given point on the surface can be classified according its principal curvatures. A point at which κ_n is equal in all directions is called an *umbilic* point; for example, every point on a sphere is an umbilic point. In the special



(a) Parabolic ($K = 0$) (b) Hyperbolic ($K < 0$) (c) Elliptic ($K > 0$)

FIGURE 3. The three non-umbilic point classifications.

case that $\kappa_n = 0$ in all directions, such as on a plane, the point is called a *flat* point. For non-umbilic points, the principal curvatures are well-defined. Figure 3 illustrates the three resulting categories. A *parabolic* point is one where a single principal curvature is 0. At an *elliptic* point both principal curvatures are positive. And at a *hyperbolic* point, one principal curvature is positive and one is negative. Note that there is a sign ambiguity present in the curvatures κ_1 and κ_2. If we flip the orientation of the surface normal, the signs of the curvatures will flip as well.

All information about the local curvature of a surface can be encapsulated in a single tensor

$$\mathbf{S} = \mathbf{G}^{-1}\mathbf{B} \tag{38}$$

This tensor is variously called the *curvature tensor*, *shape operator*, and the *Weingarten map*. It has several important properties. Firstly, its eigenvalues are the principal curvatures κ_1, κ_2. Its corresponding eigenvectors are the principal directions $\mathbf{e}_1, \mathbf{e}_2$. Because the principal curvatures of $\mathbf{S}$, we also know that the Gaussian curvature is

$$K = \det \mathbf{S} = \frac{b}{g} \tag{39}$$

and the mean curvature is

$$H = \frac{1}{2} \operatorname{tr} \mathbf{S} \tag{40}$$

8. Geodesics

Suppose that we're given a unit-speed curve $\mathbf{y}$ that lies on the surface and passes through the point $\mathbf{p}$. At this point, the curve has a unit tangent $\mathbf{t} = \dot{\mathbf{y}}$ and a unit normal $\mathbf{m} = \ddot{\mathbf{y}}$. The curvature vector $\kappa\mathbf{m}$ of this curve can be decomposed as

$$\kappa\mathbf{m} = \ddot{\mathbf{y}} = \kappa_n \mathbf{n} + \kappa_g \mathbf{s} \tag{41}$$

where $\mathbf{n}$ is the unit surface normal and $\mathbf{s} = \mathbf{n} \times \mathbf{t}$ is tangent to the surface. The curvature κ_n is the normal curvature of the surface in the direction $\mathbf{t}$. The curvature κ_g is called the *geodesic curvature*. Note that one consequence of this equation is that

$$\kappa_n = \kappa(\mathbf{m} \cdot \mathbf{n}) = \kappa \cos\phi \tag{42}$$

where ϕ is the angle between the curve's normal $\mathbf{m}$ and the surface normal $\mathbf{n}$.

A *geodesic curve* is one whose geodesic curvature κ_g is everywhere 0. For a point $\mathbf{q}$ in the vicinity of $\mathbf{p}$, the curve passing through both $\mathbf{p}$ and $\mathbf{q}$ with shortest arc length between them will be a geodesic. Therefore, geodesics are in an intuitive sense the "straight" lines intrinsic to a surface.

Part 3. Mappings

Given two surfaces M_1 and M_2, we are interested in exploring the properties of functions $f : M_1 \to M_2$ that provide a continuous mapping of points on M_1 into corresponding points on M_2. We classify mappings based on those geometric properties of the surface that they preserve. There are several classes of mappings that are commonly defined, but we are particularly interested in the following:

- *Isometric mapping* — preserves lengths.
- *Conformal mapping* — preserves angles.
- *Equiareal mapping* — preserves areas.
- *Geodesic mapping* — the image of a geodesic is a geodesic.

In the following sections, we explore some of the specific properties of these mappings. Throughout this discussion, we assume that our attention is restricted to a given patch of M_1 parameterized by the function $\mathbf{x} : E^2 \to E^3$. This induces a parameterization $\mathbf{y} : E^2 \to E^3$ of the corresponding patch of M_2 where $\mathbf{y} = f \circ \mathbf{x}$. We say that f is an *allowable* mapping if $\mathbf{y}$ meets our basic regularity requirement that $\mathbf{y}_1 \times \mathbf{y}_2 \neq 0$. The parameterizations $\mathbf{x}, \mathbf{y}$ of M_1, M_2 induce metrics $\mathbf{G}_1, \mathbf{G}_2$.

9. Isometric Mappings

Isometric mappings, or *isometries*, preserve lengths. This means that, for any curve C on M_1, the lengths of C and $f(C)$ are identical. More specifically, we can say that

$$\int_C ds_1 = \int_{f(C)} ds_2 \quad \text{for any curve } C \text{ on } M_1 \tag{43}$$

From this definition of isometry, it is fairly easy to prove that f is an isometric mapping if and only if $\mathbf{G}_1 = \mathbf{G}_2$.

Isometric mappings are a very restricted class of mappings. The requirement that $\mathbf{G}_1 = \mathbf{G}_2$ implies that the two surface patches under consideration must have identical intrinsic geometries. One easy consequence of this is that isometric surfaces must have equal Guassian curvatures at every pair of corresponding points. Thus, the only surfaces that may be mapped isometrically into the plane are the developable surfaces.

10. Conformal Mappings

As stated above, conformal mappings are those that preserve angles. Consider two curves C, D on M_1 meeting at a point $\mathbf{p}$ with angle θ. If f is a conformal mapping then their images, $f(C)$ and $f(D)$, will also meet with angle θ at $f(\mathbf{p})$.

The class of conformal mappings is much broader than the class of isometries. Specifically, the mapping f is conformal if and only if $\mathbf{G}_2 = c\mathbf{G}_1$ for some smoothly varying local scale function c. If c is constant over the entire surface, then f is in fact a *similarity mapping* and is for all practical purposes an isometry.

10.1. **In the Plane.** It is also fruitful to consider the simplest case of conformal mappings in the plane. For a variety of reasons, it is convenient to discuss such mappings in terms of complex-valued functions.

We define $f : \mathbb{C} \to \mathbb{C}$ in terms of a real part u and an imaginary part v:

$$f(z) = u(z) + iv(z) \tag{44}$$

The function f is thus a mapping of the complex plane onto itself. To simplify our discussion somewhat, let us assume that u and v are real-valued functions:

$$f(x + iy) = u(x, y) + iv(x, y) \tag{45}$$

It turns out that f is a conformal map if and only if its derivative $f' = df/dz$ exists. The requirement that f' exist is equivalent to the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{46}$$

These are the Cauchy-Riemann equations. They can also be restated in the following form:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \tag{47}$$

The Cauchy-Riemann equations provide the basic conditions for the existance of the derivative f'. Another important consequence is that if u, v satisfy the Cauchy-Riemann equations, then they also satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{48}$$

The converse is, however, not true. Functions satisfying the Laplace equation — which are called *harmonic functions* — do *not* necessarily satisfy the Cauchy-Riemann equations and are thus not necessarily conformal.

Part 4. Geometry of Discrete Manifolds

Up to this point, we have considered the geometry of differentiable manifolds. This is of course the setting in which differential geometry was developed. However, discrete mesh representations of surfaces are at least as common as differentiable surfaces in the applications of interest to us. In this section, we briefly discuss the extension of many of the classical concepts to the setting of discrete manifolds.

11. Notation and Conventions

We assume that we are dealing with a piecewise linear simplicial manifold $M = (V, E, F)$ where V, E, and F are sets of vertices, edges, and faces. Assuming that each of the vertices in V is assigned an index, we represent both edges and faces as tuples of vertex indices. Unless noted otherwise, these simplices are assumed to be *oriented*. This means that the pairs (i, j) and (j, i) both refer to the same edge connecting vertices i and j, but with opposite orientations. Similarly, the triangles (i, j, k) and (j, k, i) are equivalent, but (k, j, i) has opposite orientation.

As before, we will be concerned with the geometry of this surface when embedded in the Euclidean space E^3. We will denote the position of vertex i by $\mathbf{x}_i$. For a given oriented edge (i, j) — the edge from i to j — we define the *edge vector* $\mathbf{e}_{ij} = \mathbf{x}_j - \mathbf{x}_i$.

For a given triangle $\sigma = (i, j, k)$, we can compute its surface normal $\mathbf{n}_\sigma$ from the cross product of its edge vectors:

$$\mathbf{n}_\sigma = \frac{\mathbf{e}_{ij} \times \mathbf{e}_{ik}}{\|\mathbf{e}_{ij} \times \mathbf{e}_{ik}\|} \tag{49}$$

Mirroring the continuous case, we assume that $\|\mathbf{e}_{ij} \times \mathbf{e}_{ik}\| \neq 0$. Furthermore, it is important to note that

$$\text{Area}(\sigma) = \frac{1}{2}\|\mathbf{e}_{ij} \times \mathbf{e}_{ik}\| \tag{50}$$

For a given vertex i, let N_i denote the set of vertices adjacent to the given vertex.

$$N_i = \{j \mid (i, j) \in E\} \tag{51}$$

Similarly, let S_i denote the set of edges opposite the vertex i

$$S_i = \{(j, k) \mid (i, j, k) \in F\} \tag{52}$$

and let T_i be the set of faces incident on i.

$$T_i = \{\sigma \mid i \in \sigma \text{ and } \sigma \in F\} \tag{53}$$

12. Calculus on Simplicial Manifolds

It is important to remember that the development of a discrete differential geometry is an active area of research. There is as yet no one agreed upon framework for doing this. The development presented here is one possible — and reasonably consistent — avenue for discretizing classical notions. But there are others.

12.1. Functions and Vector Fields.

We will generally be concerned with continuous piecewise linear functions defined over the mesh. In particular, we will regard a function f as a mapping

$$f : V \to R \tag{54}$$

and will denote by f_i the value assigned by f to vertex i. This induces a value $f(\mathbf{x})$ for any point $\mathbf{x}$ on the surface, computed by linear interpolation within the triangle (i, j, k) containing $\mathbf{x}$.

As we will see, many of the operations we wish to perform on functions of this sort are themselves linear. Therefore, it will at times be convenient to identify the function f with a vector $\mathbf{f} \in R^n$ were $n = |V|$. In this case, the value of f at vertex i is simply the i-th component (i.e., f_i) of an n-vector.

Defining tangent vector fields on discrete manifolds is somewhat more difficult than in the continuous case. On a differentiable manifold, every point has a well defined tangent plane. For a triangulated manifold, this is true only of points within a triangle. A vertex has no true tangent plane, although we frequently construct an approximate tangent plane by some local averaging procedure.

Because only triangles can truly be said to have a tangent plane, we will generally restrict our discussion of tangent vectors to those which lie in a specific triangle. We will therefore view a tangent vector field $\mathbf{v}$ as a piecewise constant vector-valued function

$$\mathbf{v} : F \to R^3 \tag{55}$$

assigning a single tangent vector $\mathbf{v}_\sigma$ to each triangle $\sigma \in F$. Although it is generally convenient to think of $\mathbf{v}_\sigma$ as a 3-vector, it is a *tangent* vector, and thus always satisfies $\mathbf{v}_\sigma \cdot \mathbf{n}_\sigma = 0$. It is therefore always possible to represent $\mathbf{v}_\sigma$ as a 2-vector in some suitable orthonormal basis local to the triangle.

12.2. Gradients.

Given a piecewise linear function f defined over our mesh, we naturally want to develop some suitable notion of the derivative of f. This is easily done by a suitable definition for the gradient of f. We define the gradient ∇f to be a tangent vector field

$$\nabla f : F \to R^3 \tag{56}$$

We will denote the gradient vector within a face σ by $\mathbf{g}_\sigma$. Note that the gradient vector has two important properties. First of all, it must lie in the plane of the triangle; therefore it must be the case that $\mathbf{g}_\sigma \cdot \mathbf{n}_\sigma = 0$. It also allows as to express the restriction of f over σ as the linear function:

$$f_\sigma(\mathbf{x}) = \mathbf{g}_\sigma \cdot \mathbf{x} + b_\sigma \tag{57}$$

This definition of gradient also allows us to easily define a notion of directional derivative. Given any tangent vector $\mathbf{v}$ in the plane of the triangle σ, the directional derivative of f along $\mathbf{v}$ is:

$$\nabla_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = \mathbf{g}_\sigma \cdot \mathbf{v} \tag{58}$$

Now, suppose that we select the edge vector $\mathbf{e} = \mathbf{x}_j - \mathbf{x}_i$. The directional derivative of f along the directed edge (i, j) is simply:

$$\nabla_{\mathbf{e}} f = f_j - f_i \tag{59}$$

This leads to a particularly straightforward way of computing the gradient of f. For the triangle $\sigma = (i, j, k)$, the gradient vector $\mathbf{g}_\sigma$ is the solution to the linear system:

$$\begin{bmatrix} \mathbf{e}_{ij} \\ \mathbf{e}_{jk} \\ \mathbf{n}_\sigma \end{bmatrix} \begin{bmatrix} \mathbf{g}_\sigma \end{bmatrix} = \begin{bmatrix} f_j - f_i \\ f_k - f_j \\ 0 \end{bmatrix} \tag{60}$$

12.3. **Curl and Divergence.** The definitions provided here are based on those developed by Polthier and Preuß [14].

For a vector field in the plane, we are accustomed to analyzing its structure in terms of quantities such as its curl and divergence. We can define analogous operators for tangent vector fields on discrete meshes. As before, we assume that the vector field $\mathbf{v}$ assigns a constant vector $\mathbf{v}_\sigma$ to each triangle σ in the mesh. Furthermore, we will focus on defining the curl and divergence of such a field at the vertices of the mesh.

At vertex i, we wish to compute $\operatorname{curl}_i \mathbf{v}$.

$$\operatorname{curl}_i \mathbf{v} = \frac{1}{2} \sum_{\sigma=(j,k)\in S_i} \mathbf{v}_\sigma \cdot \mathbf{e}_{jk} \tag{61}$$

Before providing the definition of divergence, we first define a tangential rotation operator $\mathcal{R}_\sigma$. We use this operator to indicate a counter-clockwise rotation by $\pi/2$ in the tangent plane of triangle σ. Obviously, $\mathcal{R}_\sigma \mathbf{v} = \mathbf{n}_\sigma \times \mathbf{v}$. We can also write this transformation in matrix form as:

$$\mathcal{R} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{62}$$

where $\mathbf{n}_\sigma = [\, n_1 \; n_2 \; n_3 \,]^{\mathsf{T}}$.

We can now define the divergence of the vector field $\mathbf{v}$ at the vertex i as:

$$\operatorname{div}_i \mathbf{v} = \sum_{\sigma=(j,k)\in S_i} \mathbf{v}_\sigma \cdot (\mathcal{R}\mathbf{e}_{jk}) = \sum \mathbf{v}_\sigma \cdot (\mathbf{n}_\sigma \times \mathbf{e}_{jk}) \tag{63}$$

Note that this is a sum over all the triangles adjacent to vertex i. It is also a fairly simple matter to show that this definition can be rewritten as a sum over all the vertices adjacent to i:

$$\mathrm{div}_i\, \mathbf{v} = \frac{1}{2} \sum_{j \in N_i} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{v} \tag{64}$$

where α_{ij}, β_{ij} are the angles opposite the edge (i, j), as illustrated in Figure 4.

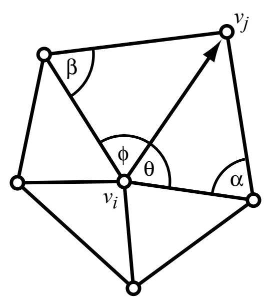


FIGURE 4. Naming angles surrounding the edge (i, j).

12.4. **Laplacian.** Given a continuous function $f : R^n \to R$, the Laplacian of f is defined to be:

$$\Delta f = \nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \tag{65}$$

The discretization of this on a simplicial 2-manifold is a mapping

$$\Delta f : V \to R \tag{66}$$

where

$$\Delta f_i = -\frac{1}{2} \sum_{j \in N_i} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i) \tag{67}$$

Note that this definition preserves the property that $\Delta f = \mathrm{div}(\nabla f)$, as in the continuous case.

For background on deriving this definition of the Laplacian, see Pinkall and Polthier [13], Duchamp *et al.* [5], and Desbrun *et al.* [2].

12.5. **Discrete Differential Forms.** A common formalism of modern calculus is the differential form. Indeed, some developments of the differential geometry of manifolds makes extensive use of differential forms[1]. In this section, we briefly outline an extension of differential forms to discrete surfaces (where, by rights, they probably ought to be called difference forms).

[1]In some older texts, you may encounter the use of the term *Pfaffian*. This is an older term, synonymous with differential form, that has fallen out of use.

A 0-form is simply a piecewise linear function $f : V \to R$, assigning scalar values to vertices.

A 1-form ω assigns a scalar value to each oriented edge of M. It must also, by definition, satisfy the property that $\omega_{ij} = -\omega_{ji}$. It is thus an antisymmetric function $\omega : E \to R$.

A 2-form assigns scalar values to oriented triangles of M. As before, it must also be antisymmetric, namely $\alpha_{ijk} = -\alpha_{kji}$.

Given a piecewise linear function f defined at the vertices of M, the differential $df : E \to R$ is a 1-form given by

$$df_{ij} = f_j - f_i \tag{68}$$

Notice that this corresponds exactly with our definition of directional derivatives given in Section 12.2. Specifically, we can see that

$$df_{ij} = \nabla_{\mathbf{e}} f \quad \text{where } \mathbf{e} = \mathbf{e}_{ij} \tag{69}$$

which is, of course, exactly what we would expect.

Given a 1-form ω, the differential $d\omega$ is a 2-form. In a given triangle (i, j, k) the value of this 2-form will be

$$d\omega_{ijk} = \omega_{ij} + \omega_{jk} + \omega_{ki} \tag{70}$$

Note this obviously implies that $d(df) = 0$. It is equally apparent that $\int_C df = 0$, for any closed cycle of edges C,

13. Discrete Curvature

As in the previous section, it is important to understand that the development of discrete notions of curvature is a research issue. Meyer *et al.* [11] provide a good discussion of the definitions that follow.

13.1. **Gaussian Curvature.** We can define the Gaussian curvature at vertex i by a direct discretization of the Gauss-Bonnet theorem:

$$K_i = 2\pi - \sum_{j \in T_i} \theta_j \tag{71}$$

13.2. **Mean Curvature.** In the continuous case, it is well known that the Laplace-Beltrami operator provides a means of computing the mean curvature normal

$$\Delta\mathbf{x} = \mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\bar{\kappa}\mathbf{n} \tag{72}$$

Given this equivalence, we can use our earlier discretization of the Laplacian over simplicial manifolds to produce an expression for the mean curvature normal at vertex i

$$\bar{\kappa}_i \mathbf{n}_i = \Delta\mathbf{x}_i = -\frac{1}{2} \sum_{j \in N_i} (\cot\alpha_{ij} + \cot\beta_{ij})(\mathbf{x}_j - \mathbf{x}_i) \tag{73}$$

Part 5. Further Reading

A comprehensive introduction to differential geometry is clearly far beyond the scope of these notes. Fortunately, there are a wide variety of books available on the subject. The classic text of Hilbert and Cohn-Vossen [7] provides an excellent introduction to the intuitive side of the subject matter with a minimum of formalism. Besl and Jain [1] give a nice overview of the essential material, and they discuss some computational techniques. For a more comprehensive and systematic treatment of the subject, I have found Kreyszig's text [8] — an expanded version of an earlier book [9] — to be fairly useful. Kreyszig's book uses the more modern tensor notation. Willmore [16] provides a fairly easy to read introduction using the somewhat dated classical notation. O'Neill [12] is a widely used and well written introductory book that uses the third major notation system, based on covariant differentiation, vector fields, and the shape operator.

Laugwitz [10] provides an admirably terse presentation of a truly impressive amount of material. Unfortunately, the notation can be a little confusing. Reading this book requires careful attention, but it's a valuable reference. Struik [15] is a fair text that uses the classical notation. It's best feature is the amount of historical background it provides. The book by do Carmo [3] seems to fairly popular and he has more recently written a companion book on Riemannian geometry [4]. You might also consider the book by Gray [6] that provides fairly extensive examples that can be used with Wolfram's *Mathematica* software.

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