Notes On Mesh Parametrization

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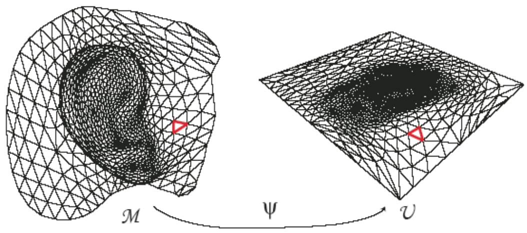

Figure 1: A mapping between 3D patch and 2D planar domain. Image taken from [1].

1 Introduction

Mesh parameterization is defined as mapping a 3D surface mesh onto a suitable target domain. Although there is no restriction on mesh topology or target domain, in general the parameterization problem is formulated as : given a 3D triangulated patch M, we want to find a piecewise linear mapping ψ between M and a planar domain $U \in \mathbf{R}^2$ (See Figure 1).

Parameterization has lots of applications in computer graphics such as texture mapping, remeshing and morphing. Texture mapping a 2D image onto a 3D mesh would previously cost lots of efforts and time for an experienced modeler to tweak the u, v texture coordinates of the mesh. By parameterizing the mesh onto a rectangular 2D domain, this becomes an easy task of mapping a 2D image onto an rectangle. Parameterizing a 3D mesh would also benefits the task of remeshing since it will be easier to redistribute the samples and to perform re-triangulation on 2D domain. Moreover, by transforming the 3D irregular mesh data into 2D regular resampled data [9], a good number of well-established algorithms which works generally on 2D regular grids could then also be effectively performed on the mesh, like JPEG compression.

There are roughly two research directions in parameterization, planar and non planar. The planar parameterization focuses on minimizing the specific stretching when mapping the 3D patch onto the 2D planar domain. The usual process of these methods is first defining an error metric which measure a specific error caused by the mapping. It then formulates the mapping as the energy minimization using the error metric and solves the system to obtain results in parameter domain.

The non-planar parameterization tries to find parameterization on the domain which is topologically equivalent to the mesh. These methods in general focus on constructing a target domain suitable for building the parameterization for the whole mesh globally.

2 Overview

2.1 Planar

Reseach works in planar parameterization deal with the problem of mapping between disk-like patch and 2D planar domain.The following section gives a high-level introduction to them.

2.1.1 Discrete Harmonic Map/Conformal Map

The early work done by Eck et al[2] finds a piecec-wise linear approximation of the harmonic map by fixing the boundary vertices and minimizing the spring harmonic energy. The energy could be expressed in the form of :
$E = \frac{1}{2} \sum_{i,j \in Edges} w_{ij} (u_i - u_j)^2$
Where u_i, u_j are the mapping results in parameter domain, and w_{ij} is the spring weighting constant defined over each

edge (i,k) using properties of original mesh (See section 3 for more detailed explanation). This configuration could be visualized as attaching a spring with different constant on each edge, and then stretching the boundary points to fixed positions. The resulting harmonic map is the minimizer of this spring energy. The minimum could be found at the critical point of function which is equal to solving a linear system defined as the zero gradient of the function,

Although it's not explicitly mentioned in the paper, the form of the energy is the same as Dirichlet energy for triangle mapping derived by Pinkall et al [16]. In Desbrun et al[1], the same energy is also used for finding the discrete conformal map(DCP) best preserving the conformality. In fact, given the same fixed boundary constraints, the mapping derived from DCP is the same as that from DHP.

2.1.2 Shape Preserving/Mean Value

A drawback in DCP/DHP is that the w_{ij} weights are defined by the cotangent function using angles from each edge's adjacent triangles. Thus if there exist obtuse triangles in the source triangulation, the value of these weights might be negative. This would then cause the mapping of a vertex to be outside its one-ring neighborhood in parameter domain, which is called the invalid embedding.

To deal with this problem, the method proposed by Floater et al[3] utilizes the fact that the form of the vertex coordinates in parameter domain could be written as the weighted combination of each vertex's one-ring neighborhood. Thus if all of the weights are defined to be positive then the mapping of each vertex would always be inside its one-ring, and thus a valid embedding. In Floater et al[3], the vertex and its one-ring are first mapped to a temporary 2D domain using the geodesic polar map. A convex triangle in the one-ring which includes the vertex could then be found in the 2D domain and the specific weights are computed using the area defined by the triangle and the vertex.

Another method by Floater et al[7] exploits the fact that harmonic functions satisfy the mean value theorem. The theorem could be seen as that for each point in the parameter domain, its harmonic function is equal to the average of the parameter values on a small circle centered at that point. The mapping is thus found using this fact to define the appropriate weights. The algebraic derivation by the author shows that these weights are independent of the radius of that circle, and could be expressed by the tangent functions using half-angles from adjacent triangles and the length of the edge. Because these tangent weights are always positive (since all half- angles of any triangles will be less than 90 degree), this method always produces a valid embedding for the mapping.

2.1.3 Discrete Authalic Map

In the recent work of Desbrun et al[1], they explore the Euler characteristic, which when defined on the Reimann manifold (differentiable manifold) could also be expressed as the integral of the Gaussian curvature. Inspired by this connection, the author extends the idea to the discrete manifold based on the fact that the sum of tip angles surrounding each vertex could be used to approximate Gaussian curvature in discrete case. The parameterization based on this metric could then be derived as the gradient of each tip angle with respect to its associated vertex. The author derives the discrete case of this angle gradient and defines the energy as Chi eneregy. The mapping resulted from minimization of Chi energy has the property of preserving the area among each vertex's one-ring neighborhood as much as possible. This property could be seen as the dual counterpart to that of DCP, which preserves the angle during mapping.

2.1.4 Discrete Natural Conformal Map

In contrast to the previous methods, which need to fix the boundary vertices in parameter domain, natural conformal map imposes the constraints on both internal vertices and boundary vertices and computes the parameterization by solving the optimization for both of them. The resulting parameterization has the boundary vertices also determined by the optimization, thus the natural boundary. The method is first proposed by Hormann et al[4] by minimizing the MIPS energy defined as the function of Dirichlet energy over the area in parameter domain. Although the resulting map preserves the conformality as much as possible, the defined energy is non-linear, which takes much more time to find the minimum than previous fixed-boundary methods.

To amend this drawback, Levy et al[6] and Desbrun et al[1] both propose an alternative to model the conformal energy minimization as a more tractable problem.

Although it is shown in Floater et al[15] and David et al[14] that these two methods are theoretically equivalent, their derivations do not form the same linear system. In Levy et al[6], inspired by Cauchy-Riemann equation, the method defines the metric to measure the violation of the equation and forms a linear least square system by summing up the energy for each triangle. On the other hand, Desbrun et al[1] derives the usual Dirichlet energy minimization for internal vertices, but also adds natural boundary constraints on boundary vertices. Because these boundary constraints include the area of triangle on the parameter domain, the resulting linear system now could not be solved independently for u, v parameters and thus is roughly 2 times as large as the system under fixed-boundary constraints. Although both methods are much faster than MIPS[4] due to the quadratic energy, they both have the drawback that folded triangles would now be possible to occur in parameter domain for obtuse triangles in original mesh.

2.2 Non-planar

While planar parameterization only focus on parameterization between a disk-like patch and a planar domain, the goal of non-planar parameterization is to generate a parameterization on a mesh of arbitrary topology at once. The desired properties for non-planar parameterization are to be globally smooth, applicable to arbitrary topology, and fast to solve. The methods differ mostly in their construction for the base domain at which the parameterization will be performed.

2.2.1 Voronoi Tiling

Eck and co-workers [2] parition the mesh into Voronoi tiles and build the Delaunay triangulation on it to construct the base domain. Discrete Harmonic Map is then computed for each triangular base domain. The resulting parameterization could then be used for resampling to generate multi-resolution models or to perform other mesh operations like mesh-compression and mesh editing in a simpler way.

2.2.2 MAPS

Instead of partition the mesh to find respective parameterization, Lee and co-workers [8] utilize the mesh simplification algorithm as the intermediate method for construting based domain. While progressively simplifying the mesh, the method also keeps track of the barycentric coordinates of the removed vertices with respect to its one-ring neighbors. Thus when simplification is complete, the parameterization of the original mesh is constructed as the linear combination of the base domain vertices. Remeshing is then operated using modified Loop subdivision on the parameter domain. Although by parameterizing on the simplified mesh, the method works for mesh of arbitrary topology, it fails to produce a smooth parameterization across the base domain triangles and thus needs to apply additional smoothing operation during remeshing.

Motivated by constructing a globally smooth paramterization, Khodakovsky and co-workers [13] use a similar method for constructing the base domain patches using simplification. However, instead of directly fixing the parameter values for vertices on patch boundary, the method fixes only the corner vertiex parameters in the base domain patches and solves parameterization for all other vertices in original mesh. This parameterization could be formulated and obtained using any existing metric to minimize the distortion. One difficulty that rises is that some of a vertex's one-ring neighbors might lie in different base domain patch, which makes the formulation useless if the one-ring of a vertex across different patches. To overcome this difficulty, a transition function is used to transform the barycentric coordinate in one patch with respect to a neighboring patch. Thus each vertex and its one-ring could be expressed in the parameter domain of the same patch, regardless of the patches they belong to. Because the transition function is defined to be linear, it won't change the linear nature of the formulation. Therefore a large linear system could then be defined over all mesh vertices, and the resulting global parameterization is obtained by solving the system.

2.2.3 Mesh Cut

Unlike the methods such as Voronoi tiling and MAPS, which need to construct a topologically equivalent base domain for the mesh, mesh cut method tends to find a

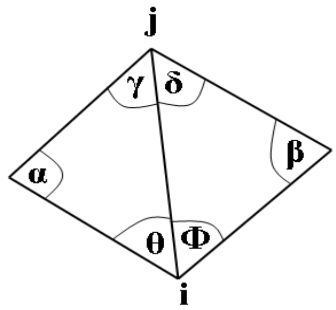

Figure 2: Angles for discrete conformal map, discrete authalic map, and mean value coordinates.

good path to cut the mesh open into a patch, and the resulting patch could then be directly parameterized using planar methods. Although this method tries to minimize the distortion of parameterizing the whole mesh by finding a good cut, it also introduces discontinuity across the cutting path. In the recent work by Gu et al[9], a mesh of arbitrary genus is cut into a disk-like patch, and the resulting parameterization is regularly resampled into an image to utilize compression and remeshing schemes. Another method proposed by Sheffer et al[10] finds a cut through high curvature vertices of the mesh to form a parameterization for better texture mapping result, reducing the distortion due to flattening the mesh.

3 Comparison of Different Metrics

Having introduced a few different metrics for parameterization, the following section gives a more detailed look into some widely used metrics and their properties in applications.

3.1 Definition

Because the parameterization based on the metrics introduced in this section could all be obtained by solving a specific linear system form, we could express the linear system in general as :

$$\sum_{j\in N(i)} w_{ij}|\mathbf{u}_i - \mathbf{u}_j| = 0$$

Where $\mathbf{u}_i, \mathbf{u}_j$ are vertex coordinates we want to find in parameter domain and w_{ij} is the weight defined by different metrics using properties on original mesh. Until further specified, these notations would be used in the section. Based on the metric used, the resulting linear system would vary and thus the u, v coordinates for each vertex in the parameter domain would also change accordingly. Therefore the formulation of these metrics differ in how their weights are defined.

3.2 Discrete Conformal Map(DCP)

Introduced by Eck et al[2], and Desbrun et al[1], this metric minimizes the Dirichlet energy and is written in the form of :

$$E_A = \sum_{edges(i,j)} \cot\alpha_{ij}|\mathbf{u}_i - \mathbf{u}_j|^2$$

Where α_{ij} and β_{ij} are corresponding angles in original mesh, and $\mathbf{u}_i, \mathbf{u}_j$ are vertex coordinates in parameter domain. See Figure 2.
Thus its critical point could be derived as :

$$\frac{\partial E_A}{\partial \mathbf{u}_i} = \sum_{j\in N(i)} (\cot\alpha_{ij} + \cot\beta_{ij})|\mathbf{u}_i - \mathbf{u}_j| = 0$$

DCP could be seen as an angle preserving mapping which minimizes the angle distortion for the interior vertices. The resulting mapping will preserve the shape but not the area of the original mesh. For example, if we map a checkboard image on the parameterization, the resulting texture mapped mesh will have the square of different sizes. (See Figure 3 and Figure 5) However, since it needs to fix the boundary vertices, the resulting triangles near the boundary would thus be distorted in both areas and angles. This method is more suitable for texture mapping a texture with highly regular pattern such as checkboard, since it will maintain the shape of the pattern in the resulting texture-mapped mesh.

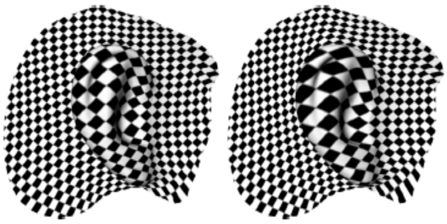

Figure 3: Comparison of a checkboard image mapping to the mesh using DCP and DAP. Left : DCP, Right : DAP. Notice the angle distortion in the DAP. Image taken from [1].

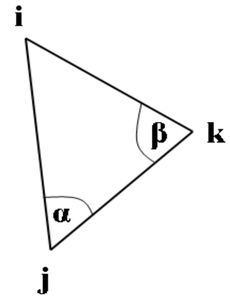

Figure 4: Angles for natural conformal map.

3.3 Discrete Authalic Map(DAP)

Also introduced by Desbrun et al [1], this metric minimizes the Chi energy and is written as the form of :

$$E_\chi = \sum_{j \in N(i)} \frac{\cot \gamma_{ij} + \cot \delta_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|^2} |\mathbf{u}_i - \mathbf{u}_j|^2$$

Where γ_{ij} and δ_{ij} are corresponding angles, $\mathbf{x}_i, \mathbf{x}_j$ are corresponding vertices in original mesh, and $\mathbf{u}_i, \mathbf{u}_j$ are vertex coordinates in parameter domain. See Figure 2. Thus its critical point could be derived as :

$$E_\chi = \sum_{j \in N(i)} \frac{\cot \gamma_{ij} + \cot \delta_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|^2} |\mathbf{u}_i - \mathbf{u}_j| = 0$$

Analogous to DCP, the method is an area preserving mapping which minizes the area distortion. Although the area of the original mesh would locally be preserved, the shape tends to be distorted since the mapping from 3D to 2D will in general generate distortion. An example of this is to map a checkboard image (see Figure 3 and Figure 5 for comparison between DCP and DAP) onto the mesh, and the resulting texture mapping will have squares whose shapes are distorted while locally having the same size.

3.4 Discrete Natural Conformal Map (DNCP)

To find a natural boundary in parameterization, the work by Desbrun et al[1] proposes another constraint imposed on each boundary vertex i in the form of :

$$\sum_{\Delta_{ijk}} \cot \alpha (\mathbf{u}_i - \mathbf{u}_j) + \cot \beta (\mathbf{u}_i - \mathbf{u}_k) = \sum_{\Delta_{ijk}} R^{90} (\mathbf{u}_k - \mathbf{u}_j)$$

Where the angles α and β are corresponding angles at k and j in triangle Δ_{ijk}, and R^{90} is a rotation by 90°. See Figure 4.
The advantage of this mapping is that only two boundary vertices need to be fixed to avoid the trivial solution of degeneracy, and thus the distortion of vertices near boundary would be less than those created by fixed boundary method. Please note that although introducing natural boundary could reduce distortion and stretching near boundary, if the boundary is ill-chosen, the resulting parameterization would still be highly stretched. Also, this method will require to solve for a linear system roughly twice as large as other fixed boundary method because the right-hand side of the equation now includes both u, v coordinates in parameter domain. Thus u, v will depends on each other when solving the system and we could not solve them separately like in the case of fixed boundary. This method in general is more suitable for finding a parameterization for a well-cut mesh, since all other fixed boundary methods will be likely to greatly stretch some features of the mesh near boundary. (See Figure 6 and Figure 7 for example)

3.5 Mean Value Coordinate

Introduced by Floater et al[7], this metric is inspired by the fact that harmonic function would satisfy the mean value theorem, which is :

Figure 5: Comparison of DCP and DAP. Left:DAP, Right:DCP. Although not obvious, there is some angle distortion in DAP mapping. Image taken from [1].

$$\mathbf{u_i} = \frac{1}{2\pi r_i}\int_{\Gamma_i} \mathbf{u}ds$$

Where u is the vertex coordinate defined on the parameter domain, Γ_i is a circle centered at u_i, and r_i is the radius of the circle. This formulation could be seen as that for each vertex u_i in the parameter domain, its value (which is the result of a harmonic function we want to approximate) is equal to the average of the parameter values on a small circle surrounding the vertex. Using this theorem as a constraint to form the linear system, the author finds a fact that the weights w_{ij} only depend on the angles of adjacent triangles rather than radius of the circle. The resulting linear system formulation is therefore written in the form of :

$$\sum_{j\in N(i)} \frac{\tan(\theta_{ij}/2)+\tan(\phi_{ij}/2)}{|\mathbf{x}_i-\mathbf{x}_j|}|\mathbf{u}_i-\mathbf{u}_j| = 0$$

Where θ_{ij} and ϕ_{ij} are corresponding angles and $\mathbf{x}_i, \mathbf{x}_j$ are vertices in original mesh.See Figure 2.

The biggest advantage of this method is that for any triangles, the tangent weights will always be positive, resulting in a valid embedding. Thus it is more suitable for implementing a global parameterization scheme where it would be more troublesome to deal with folded triangles caused by negative weights.

Table 1: Attribute summary of parameterization methods.

Method	Preseved Attribute	Always Valid	Linear System
DCP	Angle	No	$N \times N$
DAP	Area	No	$N \times N$
Mean Value	N/A	Yes	$N \times N$
DNCP	Angle	No	$2N \times 2N$

3.6 Summary

Table 1 summarizes features of the above methods. Since they vary in the preserved attribute, validness of embedding, and the size of the linear system to be solved, there is not a single method that is better than others in all aspects. One need to decide the properties of the application before choosing the appropriate method for the parameterization task.

4 Conclusion

This report makes an high-level introduction to mesh parameterization, and summarizes some of the more widely-used planar methods for mapping a 3D patch onto 2D domain. After some literature survey in recent works, we found that the research works in new metrics for parameterization seem to become gradually less than before due to the diversity and efficiency of the methods summarized in the report. However, finding a good parameterization globally for the whole mesh while maintaining the smoothness is still a challenging task, especially when the distortion measured by some metrics need to be controlled or minimized. Finding a parameterization in some base domain other than plane (like spherical parameterization proposed by Gotsman et al [12] and Praun et al[11]) also draws some interest recently. We could expect more research efforts to be made in finding a globally parameterization over arbitrary mesh that effectively minimizes some metric distortion and mantains smoothness.

References

[1] M. Desbrun, M. Meyer, and P.Alliez. Intrinsic parameterizations of surface meshes. In *Computer Graphics Forum 17, 2* , pages 209-218, 2002.

[2] M. Eck T. DeRose T. Duchamp H. Hoppe M. Lounsbbery and W. Stuetzle. Multiresolution analysis of arbitrary meshes. In *Proceedings of ACM SIGGRAPH 95* , pages 173-182.

[3] M. Floater. Parameterization and smooth approximation of surface triangulations. In *Computer Aided Geometric Design,* 14(3):231-240, April 1997.

[4] K. Hormann and G. Greiner. MIPS:An efficient global parameterization method. In *Curve and Surface Design: Saint-Malo 1999* (2000), pages 153-162, Vanderbilt University Press.

[5] P. Sander, S. Gortler, J. Snyder, and H. Hoppe. Signal-specialized parameterization. In *Pro. Eurographics Rendering Workshop* (JULY), pages 87-98.

[6] B. Levy, S. Petitjean, N. Ray, and J. Maillot. Least squares conformal maps for automatic texture atlas generation. In *Proceedings of ACM SIGGRAPH 02*, pages 362-371.

[7] M. Floater. Mean value coordinates. In *Computer Aided Geometric Design,*20(1):19-27, 2003.

[8] A. Lee, W. Sweldens, P. Schroder, L. Cowsar, and D. Dobkin. MAPS:Multiresolution adaptive parameterization of surfaces. In *Proceedings of ACM SIGGRAPH 98,* pages 95-104.

[9] X. Gu, S. Gortler, J. S., and H. Hoppe. Geometry images In *Proceedings of ACM SIGGRAPH 02,* pages 335-361.

[10] A. Shffer, and J.C. Hart. Seamster: Inconspicuous Low-Distortion Texture Seam Layout. In *Proceedings of IEEE Visualization 2002,* pages 291-298.

[11] E. Praun, and H. Hoppe. Spherical parameterization and remeshing. In *Proceedings of ACM SIGGRAPH 03,* pages 340-349.

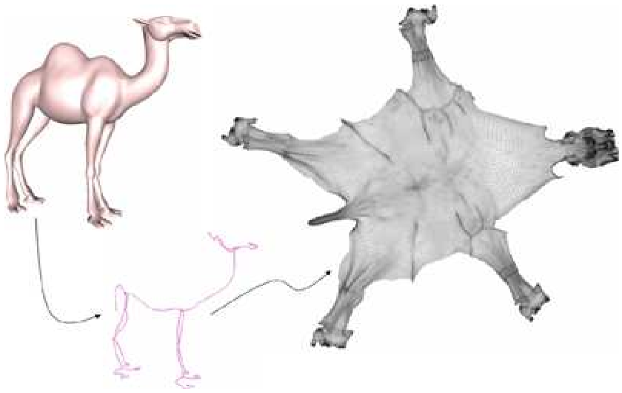

Figure 6: A camel model is cut and parameterized into 2D patch using natural conformal map. Image taken from [17].

[12] C. Gotsman, X. Gu, and A. Sheffer. Fundamentals of spherical parameterization for 3D meshes. In *Proceedings of ACM SIGGRAPH 03,* pages 358-363.

[13] A. Khodakovsky, N. Litke, and P. Schroder. Globally smooth parameterizations with low distortion. In *Proceedings of ACM SIGGRAPH 03,* pages 350-357.

[14] D. Cohen-Steiner, and M. Desbrun. Hindsight:LCSM and DNCP are one and the same.

[15] M. Floater, and K. Hormann. Surface parameterization: a tutorial and survey. In *Advances in Multiresolution for Geometric Modelling,* pages 159-284, 2004.

[16] U. Pinkall, and K. Polthier. Computing Discrete Minimal Surfaces. In *Experimental Mathematics 2*, 1 (1993), pages 15-36.

[17] Alliez, P., Colin De Verdiere., Deviller, O., and Isenburg, M. Isotropic Surface Remeshing. In *Proceedings of Shape Modeling International Conference 2003.*, pages 49-59.

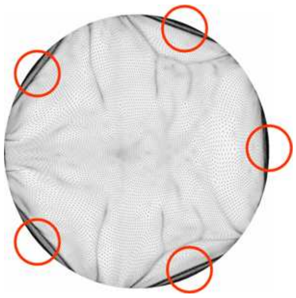

Figure 7: The same camel model in Figure 6 is parameterized using fixed-boundary method. Notice the distortion of head and legs near boundary. Image taken from [17].