Surface Simplification Using Quadric Error Metrics

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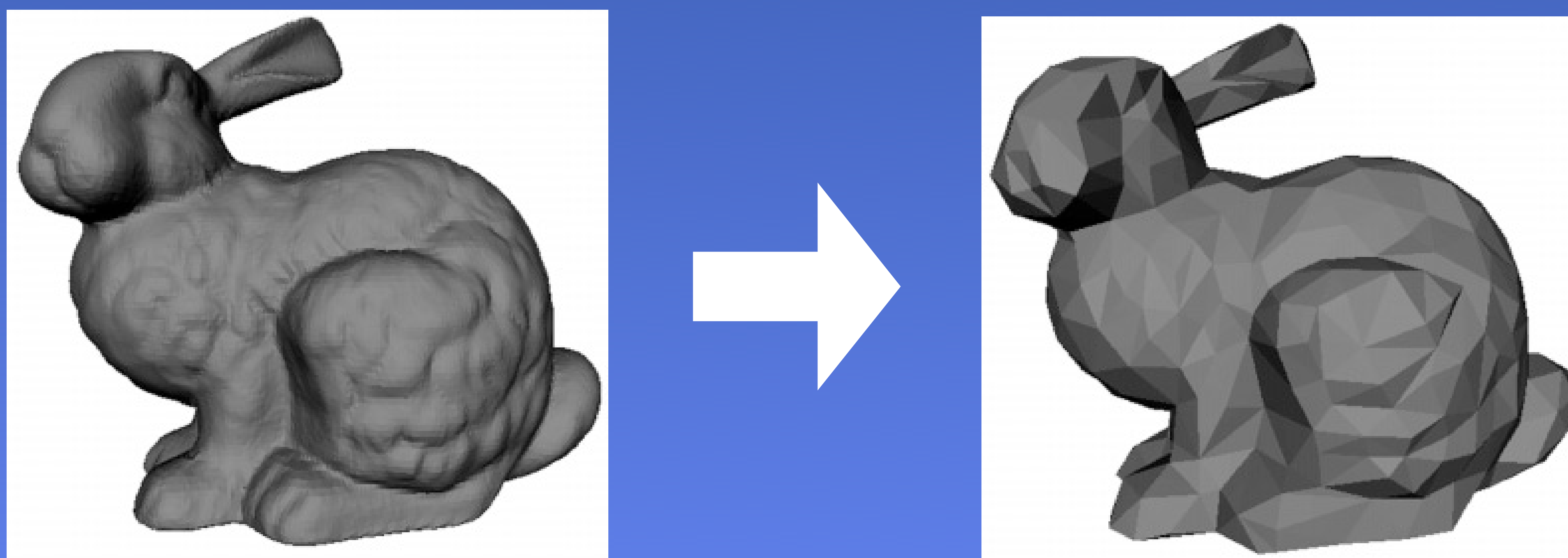
Presented by Jerry O. Talton III

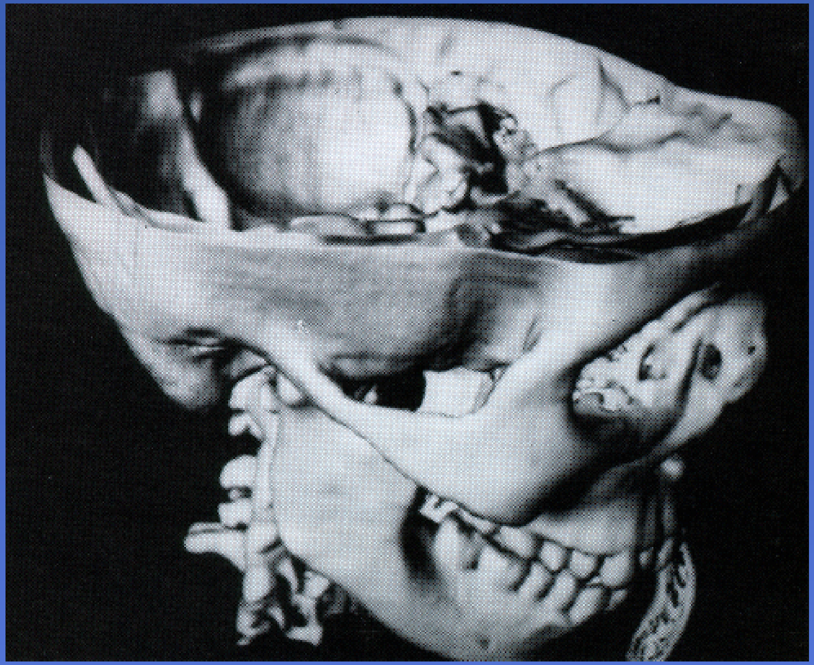
The Basic Idea

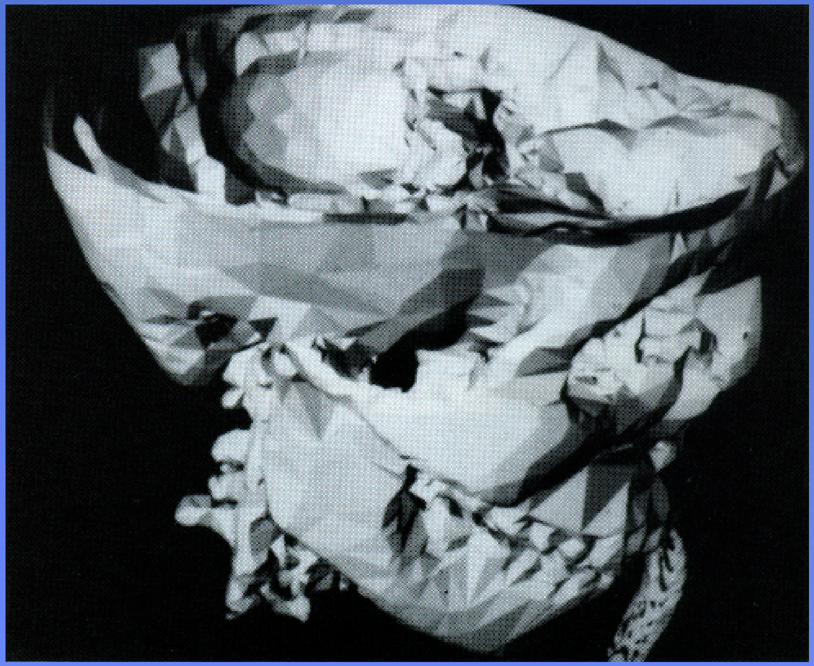


Vertex Decimation: [Schroeder92]

- Classify vertices as simple, complex, boundary, interior edge, or corner vertex.
- Iteratively remove vertices that meet some decimation criteria.
- Triangulate resulting holes.

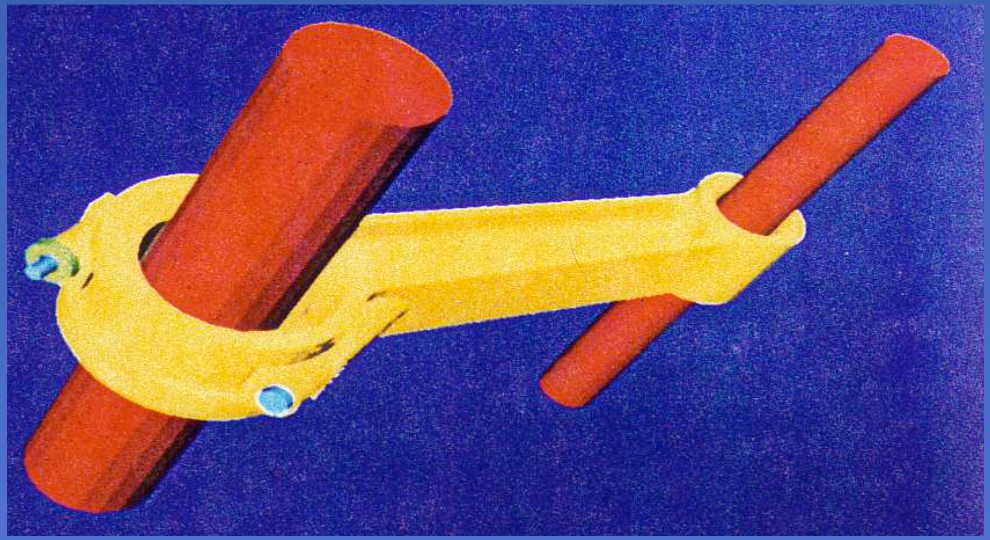
Restricted to manifold surfaces.
Carefully preserves topology.

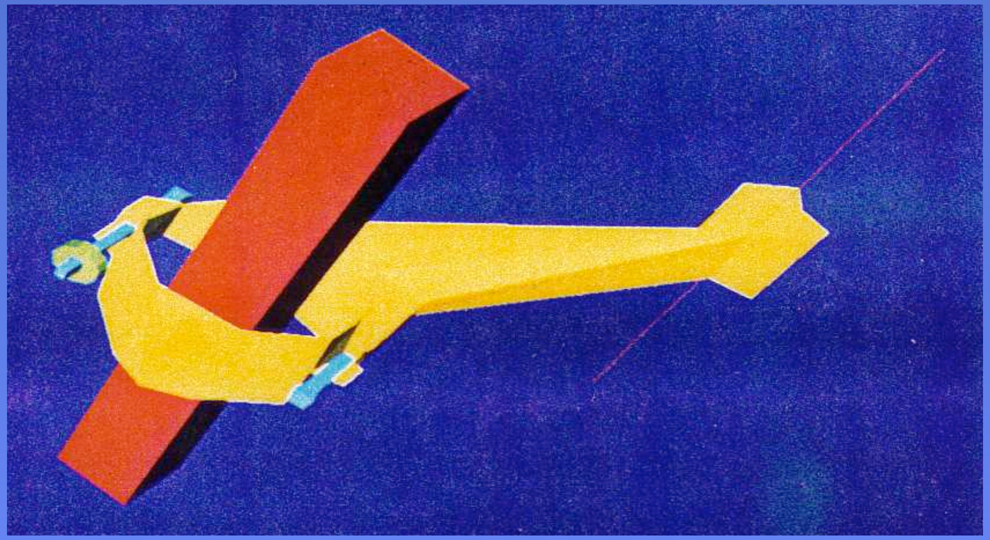




Vertex Clustering: [Rossignac92]

- Weight vertices based on perceptual importance.
- Create bounding box and subdivide into grid.
- Perform weighted clustering of vertices in each cell.





Very fast.
Works on non-manifold geometry.
May drastically alter topology.
Visually unappealing.
Difficult to produce models with N faces.

Iterative Edge Contraction: [Hoppe96] (and others)

- Define the cost of contracting an edge.
- Iteratively contract the edge with lowest cost.

High quality results.
Cost functions can be complex.
Can close holes.
Can't join disconnected components.

The Solution:

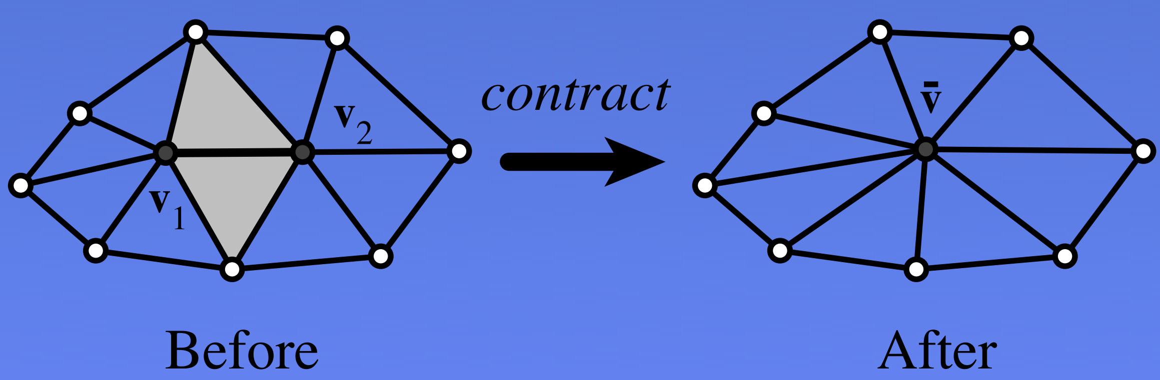
Iterative Pair Contraction with the Quadric Error Metric

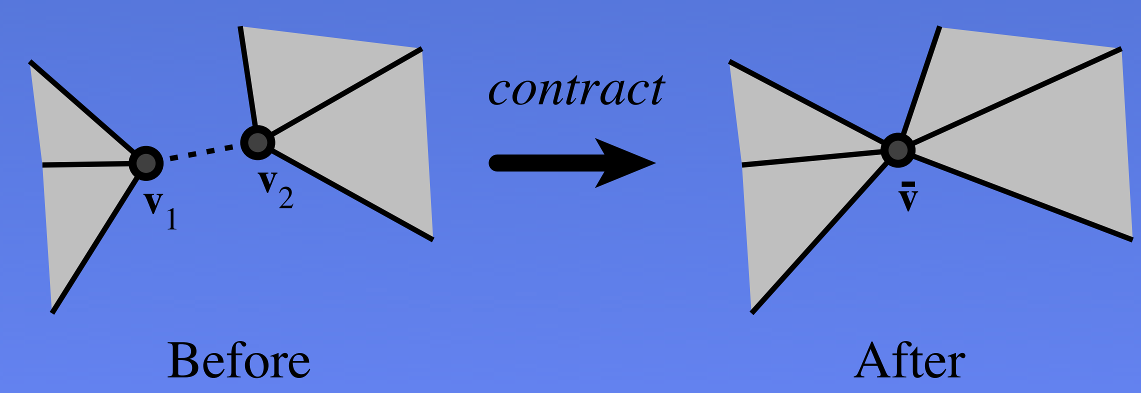
- Works on non-manifold geometry
- Supports aggregation
- Can be implemented efficiently
- Produces high quality approximations

Iterative Pair Contraction

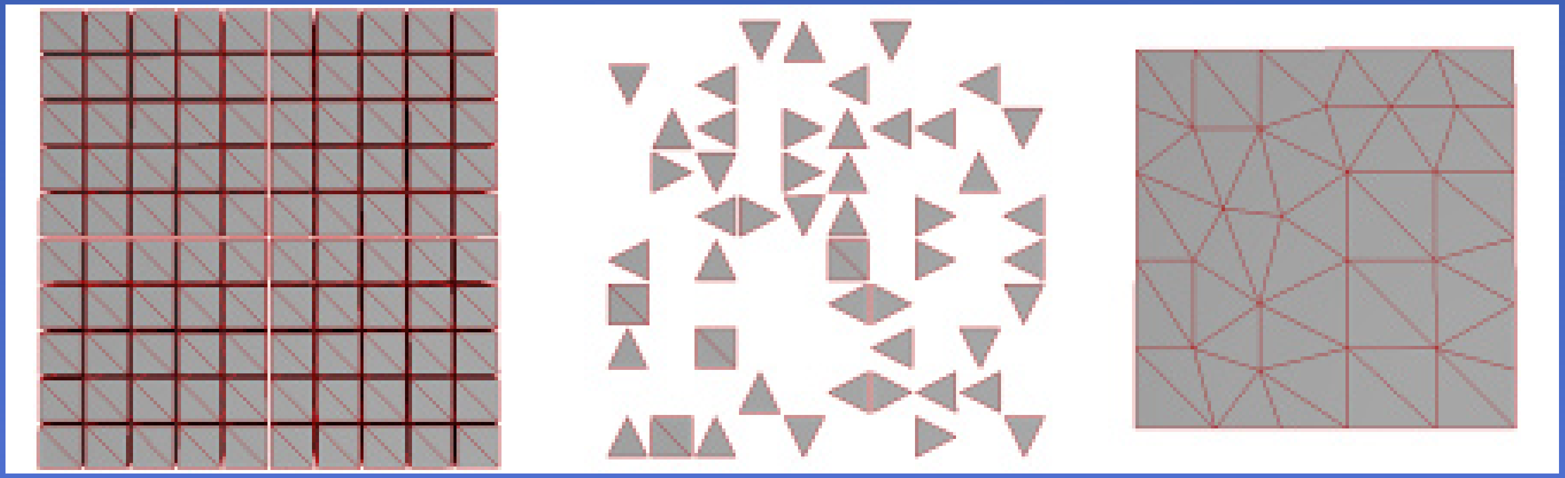
A pair of vertices $(\mathbf{v_1}, \mathbf{v_2})$ are **valid** for contraction if:

1. $(\mathbf{v_1}, \mathbf{v_2})$ is an edge, *or*
2. $||\mathbf{v_1} - \mathbf{v_2}|| < t$ for some threshold t





Benefits of Pair Contraction



- Can join unconnected components
- Can result in much nicer approximations

Error Metric

[Ronfard96] suggested the following:

- Each vertex is the intersection of a set of planes.
- Define the error at a vertex to be the sum of the squared distances to its planes.

$$\Delta(\mathbf{v}) = \Delta([v_x\ v_y\ v_z\ 1]^T) = \sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{p}^T \mathbf{v})^2$$

Where $\mathbf{p} = [a\ b\ c\ d]^T$ represents the plane $ax + by + cz + d = 0$

with $a^2 + b^2 + c^2 = 1$

Error Metric (2)

$$
\begin{aligned}
\Delta(\mathbf{v}) &= \sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{p}^T \mathbf{v})^2 \\
&= \sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{p}^T \mathbf{v})^T (\mathbf{p}^T \mathbf{v}) \\
&= \sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{v}^T \mathbf{p})(\mathbf{p}^T \mathbf{v}) \\
&= \sum_{\mathbf{p} \in planes(\mathbf{v})} \mathbf{v}^T (\mathbf{p}\mathbf{p}^T) \mathbf{v} \\
&= \mathbf{v}^T \left(\sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{p}\mathbf{p}^T) \right) \mathbf{v}
\end{aligned}
$$

Error Metric (3)

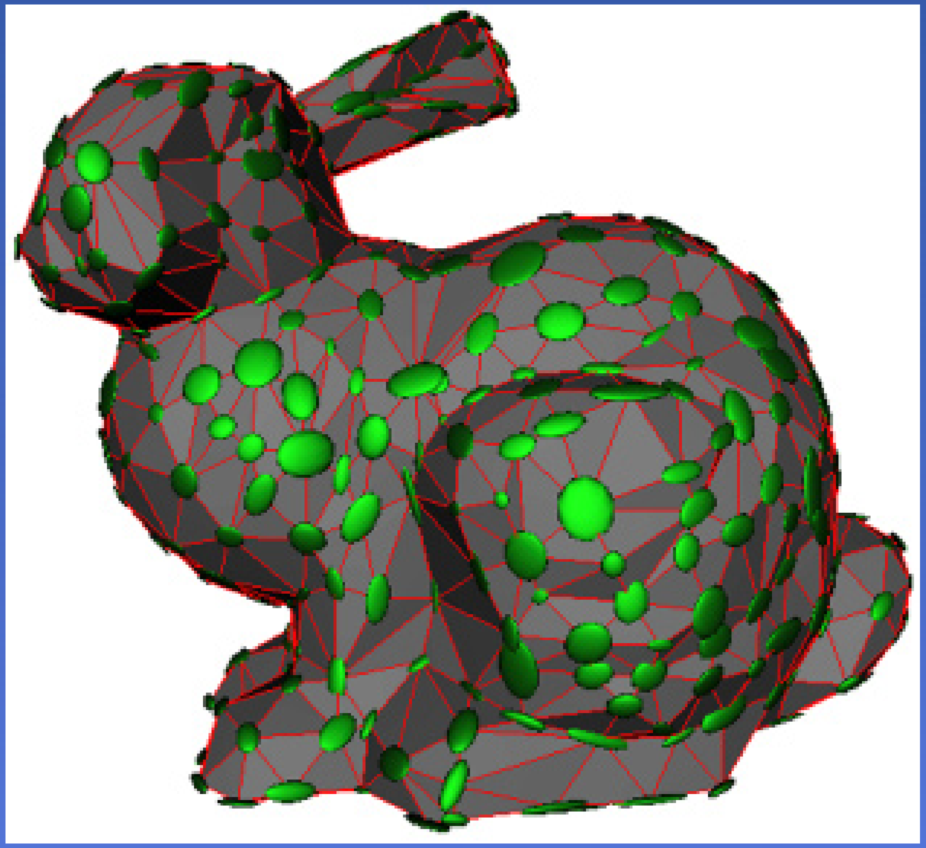
$$\Delta(\mathbf{v}) = \mathbf{v}^T \left(\sum_{\mathbf{p} \in planes(\mathbf{v})} (\mathbf{p}\mathbf{p}^T) \right) \mathbf{v}$$

$$= \mathbf{v}^T \left(\sum_{\mathbf{p} \in planes(\mathbf{v})} \mathbf{K_p} \right) \mathbf{v}$$

Where $\mathbf{K_p} = \mathbf{p}\mathbf{p}^T = \begin{bmatrix} a^2 & ab & ac & ad \\ ba & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$

$\mathrm{K_p}$ is the **fundamental error quadric.**

Error Metric (4)



- For each vertex $\mathbf{v_i}$ store a symmetric 4x4 matrix $\mathbf{Q_i}$.
- For a given contraction $(\mathbf{v_1}, \mathbf{v_2}) \rightarrow \bar{\mathbf{v}}$, let $\bar{\mathbf{Q}} = \mathbf{Q_1} + \mathbf{Q_2}$.
- The matrices $\mathbf{Q_i}$ are called quadrics because the level sets of $\Delta(\mathbf{v}) = \epsilon$ form quadric surfaces (usually ellipsoids).

More on Quadrics

$$\mathbf{v_h} = [v_x \; v_y \; v_z \; 1]^T \quad \mathbf{p} = [a \; b \; c \; d]^T$$

$$\begin{aligned} D^2(\mathbf{v_h}) = (\mathbf{p}^T \mathbf{v_h})^2 &= (\mathbf{n}^T \mathbf{v} + d)^2 \quad \text{where} \quad \mathbf{n} = [a \; b \; c]^T \\ &= (\mathbf{v}^T \mathbf{n} + d)(\mathbf{n}^T \mathbf{v} + d) \\ &= (\mathbf{v}^T \mathbf{n}\mathbf{n}^T \mathbf{v} + 2d\mathbf{n}^T \mathbf{v} + d^2) \\ &= (\mathbf{v}^T (\mathbf{n}\mathbf{n}^T) \mathbf{v} + 2(d\mathbf{n})^T \mathbf{v} + d^2) \end{aligned}$$

$$\mathbf{X} = \mathbf{n}\mathbf{n}^T = \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \quad \mathbf{y} = d\mathbf{n} = [da \; db \; dc]^T \quad z = d^2$$

More on Quadrics (2)

$$\mathbf{Q} = \begin{bmatrix} a^2 & ab & ac & ad \\ ba & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix} = Q(\mathbf{X}, \mathbf{y}, z)$$

$$\mathbf{X} = \mathbf{n}\mathbf{n}^T = \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \quad \mathbf{y} = d\mathbf{n} = [da\ db\ dc]^T \quad z = d^2$$

$$\Delta(\mathbf{v}) = \mathbf{v}^T\mathbf{Q}\mathbf{v} = \mathbf{v}^T\mathbf{X}\mathbf{v} + 2\mathbf{y}^T\mathbf{v} + z$$

Performing Contractions

To perform a contraction $(\mathbf{v_1}, \mathbf{v_2}) \to \bar{\mathbf{v}},$ we must find $\bar{\mathbf{v}}.$

Specifically, we want $\nabla(\Delta(\bar{\mathbf{v}})) = 0.$

$$\nabla(\Delta(\bar{\mathbf{v}})) = 2\mathbf{X}\bar{\mathbf{v}} + 2\mathbf{y}$$

$$2\mathbf{X}\bar{\mathbf{v}} + 2\mathbf{y} = 0 \implies \bar{\mathbf{v}} = -\mathbf{X}^{-1}\mathbf{y}$$

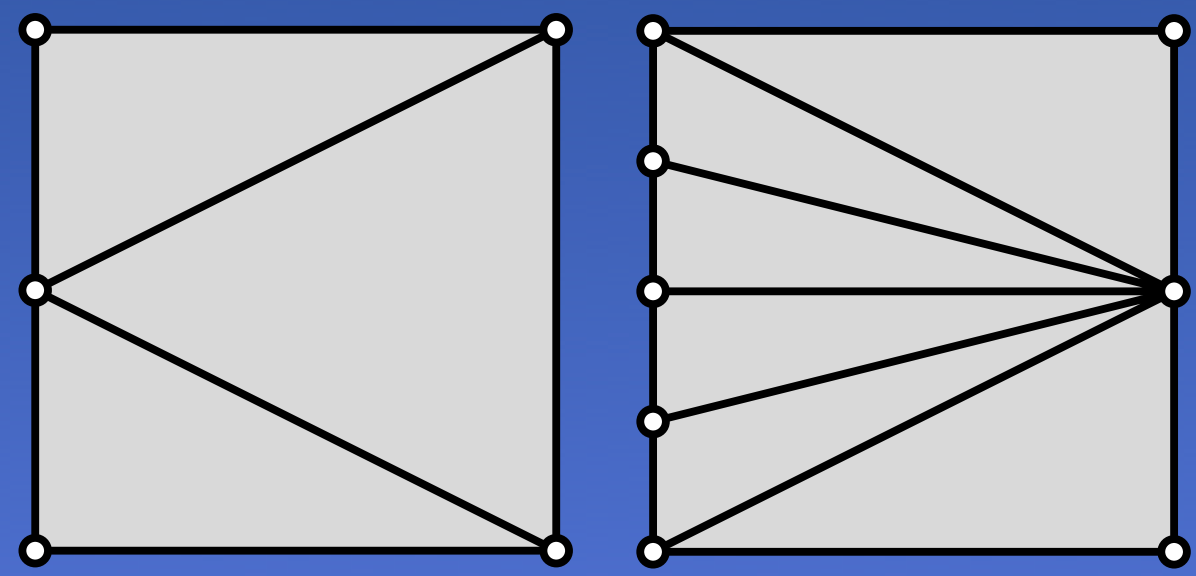
The associated minimum error is:

$$\Delta(\bar{\mathbf{v}}) = \mathbf{y}^T\bar{\mathbf{v}} + z = -\mathbf{y}^T\mathbf{X}^{-1}\mathbf{y} + z$$

Algorithm Summary

- Compute initial quadrics for each vertex.
- Select all valid pairs.
- Compute the optimal contraction target for each pair and let its associated error be the **cost** of the contraction.
- Place all pairs in a keyed heap on cost with the minimum cost pair at the top.
- Iteratively remove the pair with least cost from the heap, contract the pair, and update the cost of all valid pairs involving this contracted vertex.

Additional Details

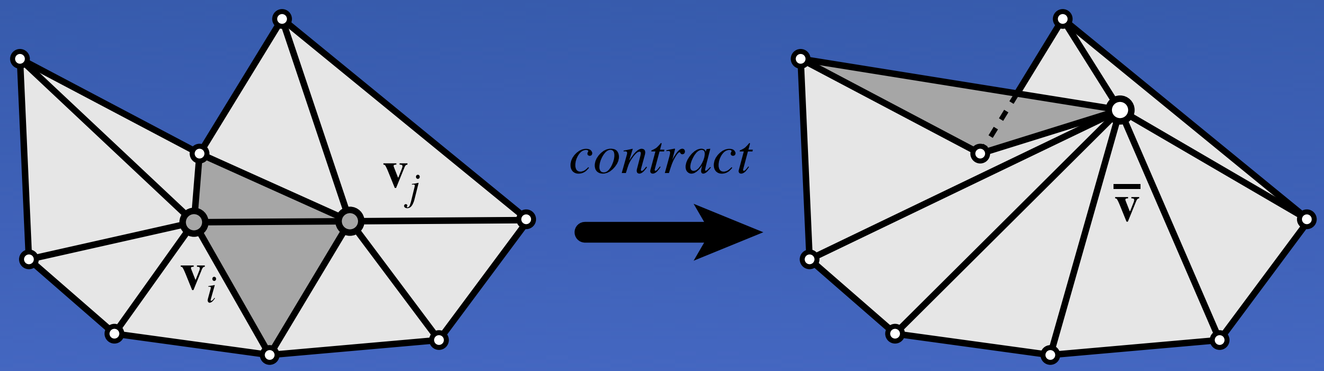


- As proposed in the paper, the algorithm is very sensitive to tessellation.
- In practice, weight each quadric according to area as in [Garland99].

Additional Details (2)

- When we wish to preserve boundaries, we can create perpendicular planes to boundary edges.
- Then, weight the associated fundamental quadrics appropriately to penalize movement away from the boundary.

Additional Details (3)



- Contractions may invert the mesh.
- The paper proposes penalizing contractions where the normal of a face changes by more than some threshold value.
- A better solution is described in [Garland99], which defines the region the contracted vertex may occupy without causing foldover.

Additional Details (4)

$$\bar{\mathbf{v}} = -\mathbf{X}^{-1}\mathbf{y}$$

- Computing inverses is bad: use Cholesky decomposition (since $\mathbf{X}$ is positive semidefinite, by construction).
- What if $\mathbf{X}$ is singular?
  - Can use SVD to project vertex onto the solution space.
  - In practice, look along line between source vertices or just pick whichever source vertex minimizes the error.

Additional Details (5)

$$\bar{\mathbf{v}} = -\mathbf{X}^{-1}\mathbf{y}$$

$$\Longleftrightarrow$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_k & b_k & c_k \\ & \vdots & \\ a_n & b_n & c_n \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_k \\ \vdots \\ d_n \end{pmatrix}$$

More on Stability: [Ju02]

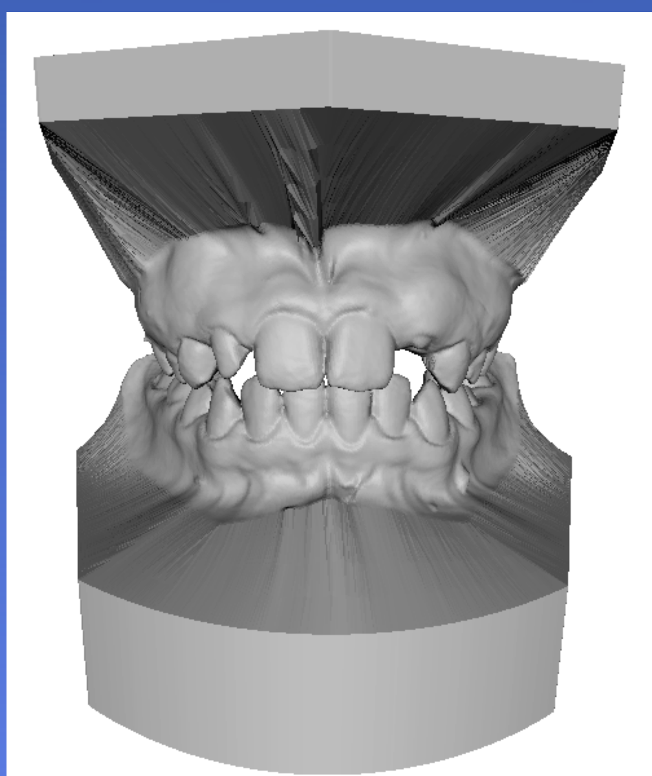
Evaluating $\Delta(\mathrm{v})$ as proposed not stable with floats.

Compute a sequence of givens rotations $\mathbf{G}$ s.t.:

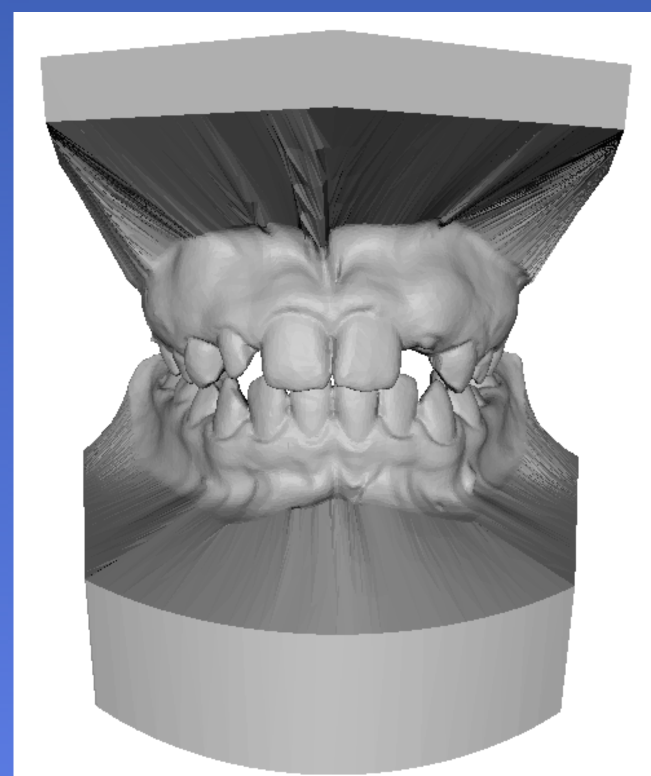
$$\mathbf{G}(\mathbf{X}\ \mathbf{y}) = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{X}} & \hat{\mathbf{y}} \\ 0 & z \\ 0 & 0 \\ \dots & \dots \end{pmatrix}$$

$$\bar{\mathbf{v}} = -\mathbf{X}^{-1}\mathbf{y} = -\hat{\mathbf{X}}^{-1}\hat{\mathbf{y}}$$

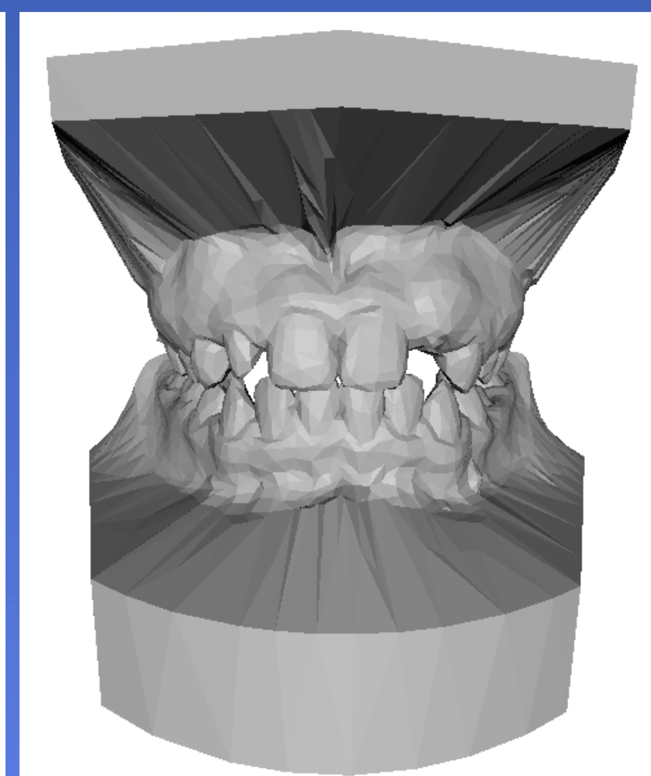
Results



424,376



60,000



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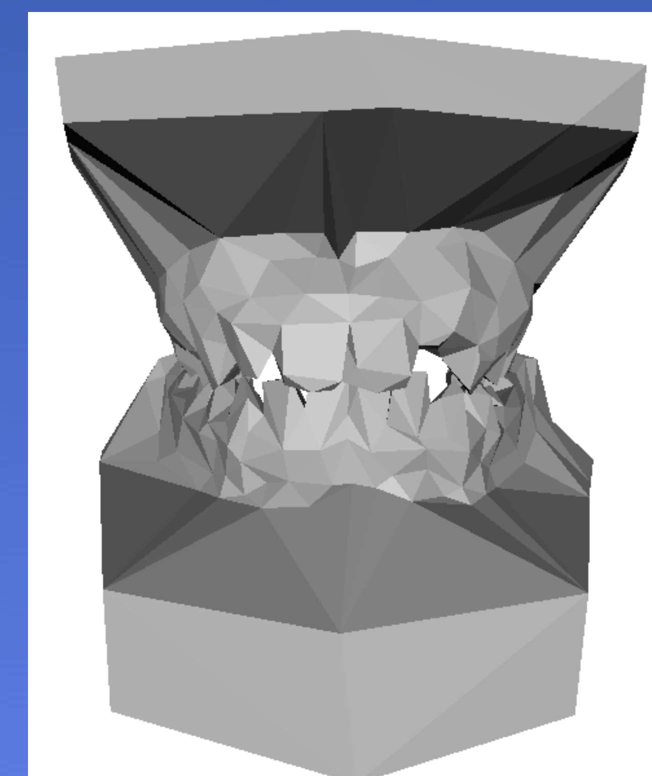
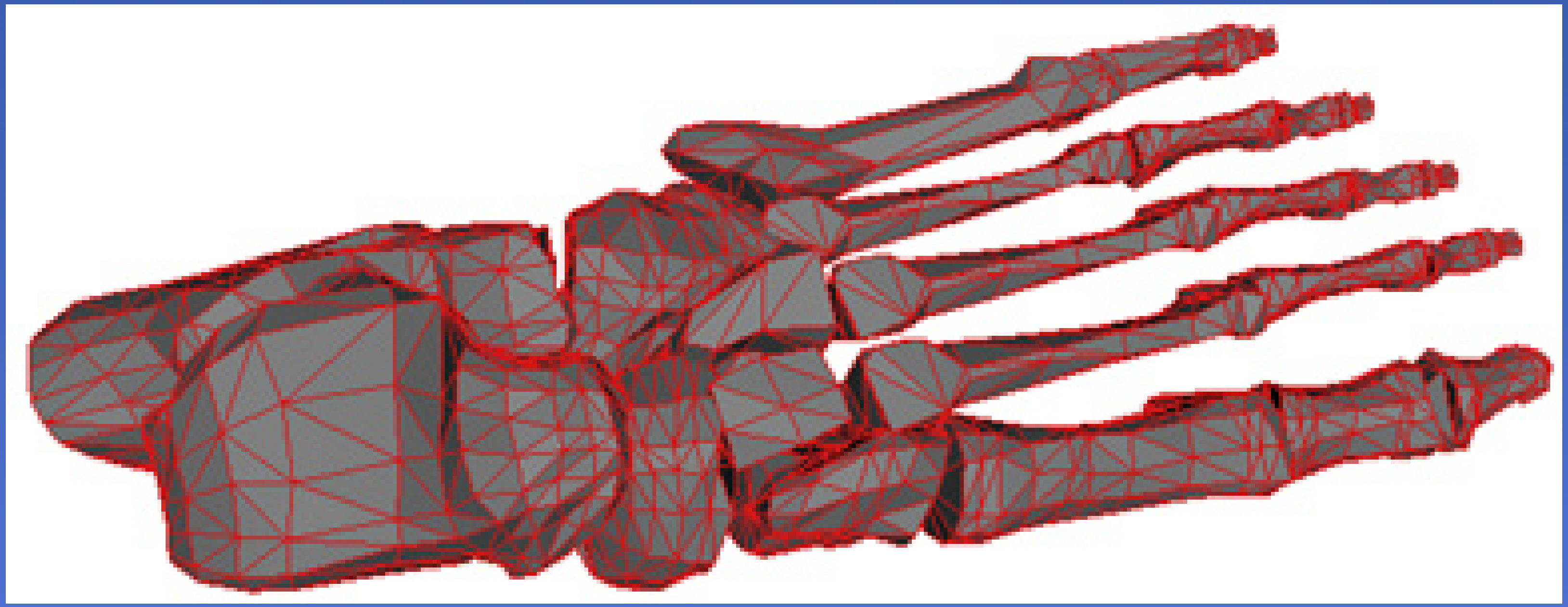
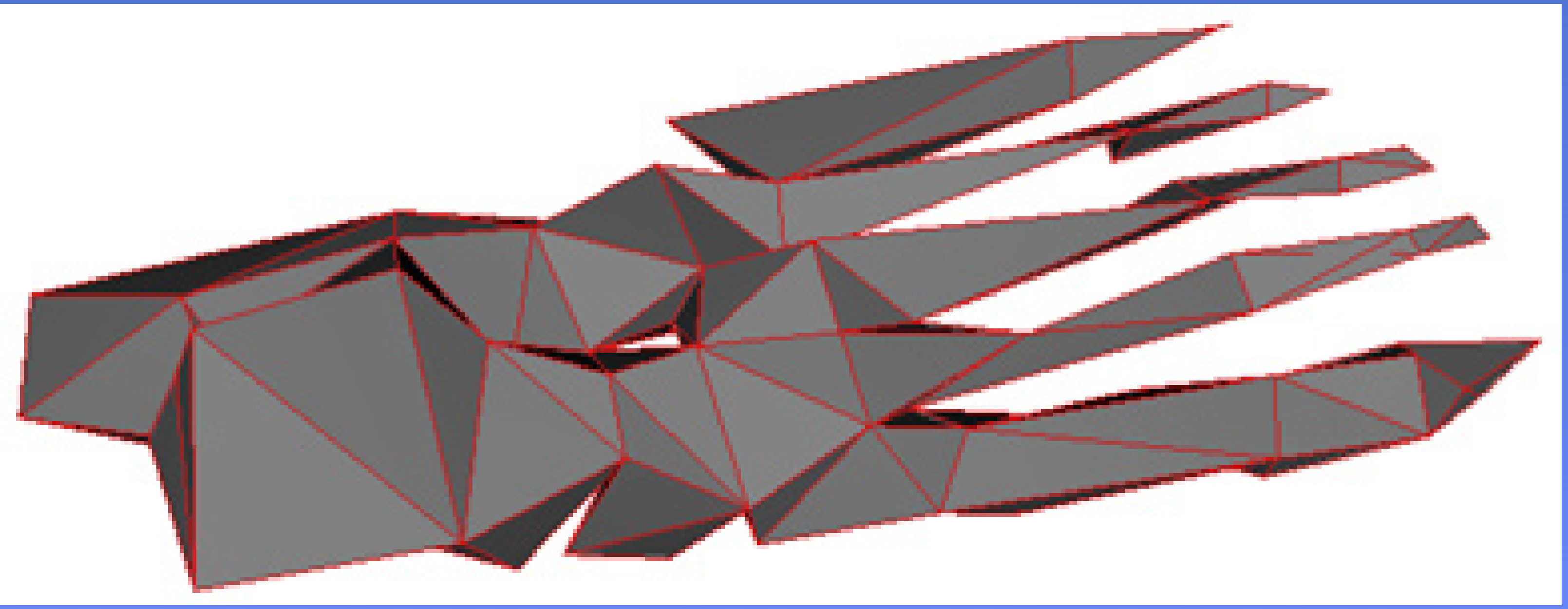


1,000

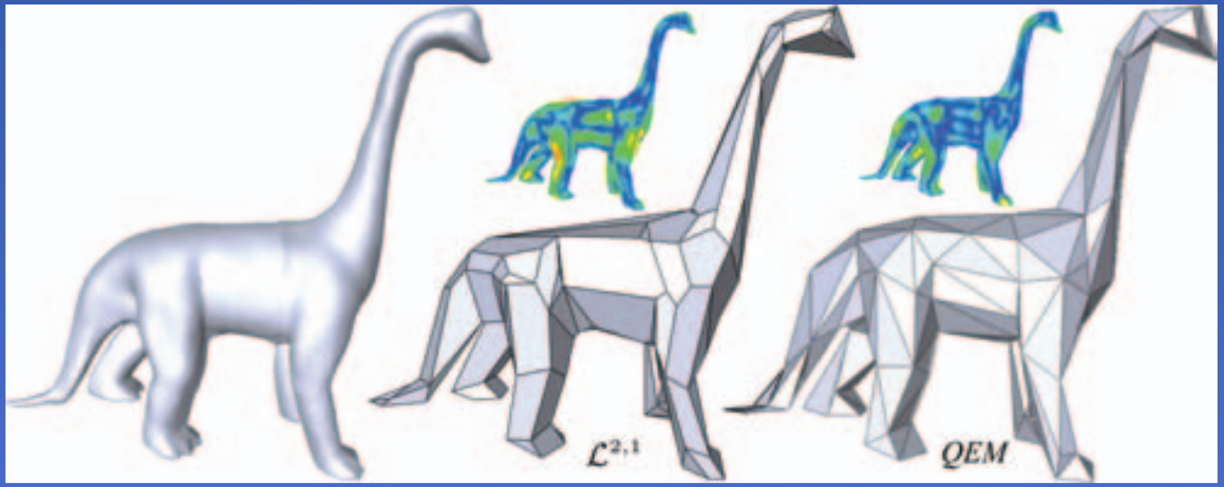
Results

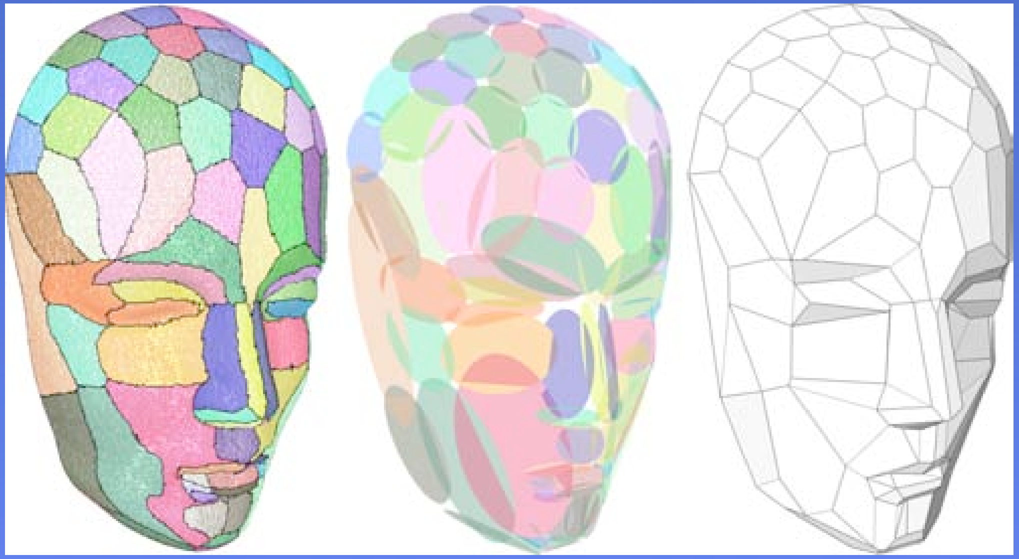




Variational Shape Approximation: [Cohen-Steiner04]

- Formulate surface simplification as an optimization problem.
- Use clustering to fit local *shape proxies* to surface.
- Use these proxies to produce approximating surfaces.





Better approximation than QER.
Much slower than QER.

References

- M. Garland and P. Heckbert. Surface simplification using quadric error metrics. In Proceedings of SIGGRAPH 97, pp. 209–216, August 1997.
- W. Schroeder, J. Zarge, and W. Lorenson. Decimation of triangle meshes. Proceedings of SIGGRAPH 92, pp.65–70, August 1992.
- H. Hoppe. Progressive meshes. In Proceedings of SIGGRAPH 96, pp. 99–108, August 1996.
- J. Rossignax and P. Borrel. Multi-resolution 3D approximations for rendering complex scenes. In Modeling in Computer Graphics: Methods and Applications, pp. 455-465, 1993.
- Michael Garland. Quadric-Based Polygonal Surface Simplification. Ph.D. dissertation, Computer Science Department, Carnegie Mellon University, CMU-CS-99-105, May 1999.
- T. Ju, F. Losasso, S. Schaefer, and J. Warren. Dual contouring of Hermite data. In Proceedings of SIGGRAPH 2002, pp. 339–346, 2002.
- D. Cohen-Steiner, P. Alliez, and M. Desbrun. Variational shape approximation. In Proceedings of SIGGRAPH 04, pp. 905–914, August 2004.